

# Fermion Fields in the (Non)Symmetric Kaluza–Klein Theory

M. W. Kalinowski

Bioinformatics Laboratory, Medical Research Centre, Polish Academy of Sciences,  
02-106 Warszawa, Poland

e-mail: markwkal@bioexploratorium.pl, mkalinowski@imdik.pan.pl

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## Abstract

The paper is devoted to the unification of fermions within Nonsymmetric Kaluza–Klein Theories. We obtain a Lagrangian for fermions in Non-Abelian Kaluza–Klein Theory and Non-Abelian Kaluza–Klein Theory with spontaneous symmetry breaking and Higgs’ mechanism. We get also Lagrangian for fermion in our approach for bosonic GSW (Glashow–Salam–Weinberg) model. We get Yukawa-type terms and mass terms coming from higher dimensions. We consider 1/2-spin fields and also 3/2-spin fields.

## 1 Introduction

The aim of the paper is to develop a formalism of the Nonsymmetric Kaluza–Klein Theory to include fermion fields getting unification of Yukawa coupling. A novel approach consists in using new kind of gauge derivatives in the Nonsymmetric Kaluza–Klein Theory and an expansion of spinor fields in zero modes of the group manifold. In the case of the Nonsymmetric Kaluza–Klein Theory with spontaneous symmetry breaking and Higgs’ mechanism we expand spinor fields in harmonics on  $H = G/G_0$  or  $S^2$  getting mass terms and Yukawa-type couplings. The use of zero modes on a group manifold allows us to avoid Planck’s mass terms for fermions.

In the paper we consider fermions in the Nonsymmetric Kaluza–Klein Theory. We suppose that fermions belong to fundamental representation of generalized Lorentz groups  $SO(1, n + 3)$  ( $Spin(1, n + 3)$ ),  $SO(1, n + n_1 + 3)$  ( $Spin(1, n + n_1 + 3)$ ) or  $SO(1, 19)$  ( $Spin(1, 19)$ ). We consider 0-form spinor fields (i.e. spinor fields in the usual sense) and 1-form spinor fields. Due to this approach we get 1/2-spin fields and also 3/2-spin fields in our theory.

The Nonsymmetric Kaluza–Klein (Jordan–Thiry) Theory (a real version) has been developed in the past (see Refs [1]–[5]). The theory unifies gravitational theory described by NGT (Nonsymmetric Gravitational Theory, see Ref. [6]) and Yang–Mills’ fields (also electromagnetic field). In the case of the Nonsymmetric Jordan–Thiry Theory this theory includes scalar field. The Nonsymmetric Kaluza–Klein Theory can be obtained from the Nonsymmetric Jordan–Thiry Theory by simply putting this scalar field to zero. In this way it is a limit of the Nonsymmetric Jordan–Thiry Theory.

The Nonsymmetric Jordan–Thiry Theory has several physical applications in cosmology, e.g.: (1) cosmological constant, (2) inflation, (3) quintessence, and some possible relations to the dark matter problem. There is also a possibility to apply this theory to an anomalous acceleration problem of Pioneer 10/11 (see Refs [7], [8]).

Simultaneously the theory unifies gravity with gauge fields in a nontrivial way via geometrical unifications of two fundamental invariance principles in Physics: (1) the coordinate invariance principle, (2) the gauge invariance principle. Unification on the level of invariance principles is more important and deeper than on the level of interactions for from invariance principles we get conservation laws (via the Noether theorem). In some sense Kaluza–Klein theory unifies the energy-momentum conservation law with the conservation of a color (isotopic) charge (an electric charge in an electromagnetic case).

This unification has been achieved in higher than four-dimensional world, i.e.  $(n + 4)$ -dimensional, where  $n = \dim G$ ,  $G$  is a gauge group for a Yang–Mills’ field, which is a semisimple Lie group (non-Abelian). In an electromagnetic case we have  $G = U(1)$  and a unification is in 5-dimensional world (see also [9]). The unification has been achieved via a natural nonsymmetric metrization of a fiber bundle. This metrization is right-invariant with respect to an action of a group  $G$ . We present also an Hermitian metrization of a fiber bundle in two versions: complex and hypercomplex. The connection on a fiber bundle of frames over a manifold  $P$  (a bundle manifold) is compatible with a metric tensor (nonsymmetric or Hermitian in complex or hypercomplex version). In the case of  $G = U(1)$  the geometrical structure is biinvariant with respect to an action of  $U(1)$ , in a general non-Abelian case this is only right-invariant.

The unification is nontrivial for we can get some additional effects unknown in conventional theories of gravity and gauge fields (Yang–Mills’ or electromagnetic field). All of these effects, which we call *interference effects* between gravity and gauge fields are testable in principle in experiment or in an observation. The formalism of this unification has been described in Refs [1]–[5], [9] (without Hermitian versions).

It is possible to extend the Nonsymmetric (non-Abelian) Kaluza–Klein Theory to the case of a spontaneous symmetry breaking and Higgs’ mechanism (see Ref. [1]) by a nontrivial combination of Kaluza principle (Kaluza miracle) with dimensional reduction procedure. This consists in an extension of a base manifold of a principal fiber bundle from  $E$  (a space-time) to  $V = E \times M$ , where  $M = G/G_0$  is a manifold of classical vacuum states.

The Nonsymmetric Kaluza–Klein Theory is an example of the geometrization of fundamental interaction (described by Yang–Mills’ and Higgs’ fields) and gravitation according to the Einstein program for geometrization of gravitational and electromagnetic interactions. It means an example to create a Unified Field Theory. In the Einstein program we have to do with electromagnetism and gravity only. Now we have to do with more degrees of freedom, unknown in Einstein times, i.e. GSW (Glashow–Salam–Weinberg) model, QCD, Higgs’ fields, GUT (Grand Unified Theories). Moreover, the program seems to be the same.

We can paraphrase the definition from Ref. [10]: *Unified Field Theory: any theory which attempts to express gravitational theory and fundamental interactions theories within a single unified framework. Usually an attempt to generalize Einstein’s general theory of relativity alone to a theory of gravity and classical theories describing fundamental interactions.* In our case this single unified framework is a multidimensional analogue of geometry from Einstein Unified Field Theory (treated as generalized gravity) defined on principal fiber bundles with base manifolds:  $E$  or  $E \times V$  and structural groups  $G$  or  $H$ . Thus the definition from an old dictionary (paraphrased by us) is still valid.

Some ideas on geometrization and unification of fundamental physical interactions can be found in Ref. [11].

Summing up, Nonsymmetric Kaluza–Klein Theory connects old ideas of unitary field theories (unified field theories, see Refs [12, 13] for a review) with modern applications. This is a geometrization and unification of a bosonic part of four fundamental interactions (see also Ref. [14]).

Let us give some sketch of the development of the Nonsymmetric Kaluza–Klein (Jordan–Thiry) Theory. In Ref. [4] we introduced the theory in the electromagnetic case. This case has been developed in details in Ref. [9]. Ref. [3] is devoted to a general non-Abelian case. In Ref. [1] one can find also non-Abelian theory with spontaneous symmetry breaking and Higgs’ mechanism. The further development of non-Abelian theory and spontaneous symmetry breaking with Higgs’ mechanism can be found in Ref. [14]. The idea of a dielectric model of color confinement in the Nonsymmetric Kaluza–Klein Theory has been introduced in Ref. [2]. This has been developed in Ref. [14]. The further development of a symmetry breaking in the Nonsymmetric Kaluza–Klein Theory (a hierarchy of a symmetry breaking) can be found in Ref. [15]. Application of the scalar field in the theory can be found in Refs [7], [8] for a Pioneer 10/11 anomaly acceleration. Some applications of this field in cosmology can be found in Ref. [16]. Ref. [16] gives some cosmological models with Higgs’ field and a scalar field  $\Psi$  (as a quintessence field). (Using the same notation  $\Psi$  for a scalar field in the Nonsymmetric Kaluza–Klein (Jordan–Thiry) Theory and for spinor field  $\Psi$  or even for spinor valued 1-form  $\Psi$  cannot cause any confusion.) It deals in great details with the cosmological evolution of Higgs’ field in the framework of cosmological models with this scalar field  $\Psi$  for Nonsymmetric Kaluza–Klein (Jordan–Thiry) Theory. Section 2 of the paper is devoted to some details of the Nonsymmetric Kaluza–Klein Theory with (and without) spontaneous symmetry breaking and Higgs’ mechanism. Section 3 gives many details of application of the theory to the bosonic part of GSW (Glashow–Salam–Weinberg) model. The application achieves several important issues:

- 1° mass spectrum of  $W^\pm, Z^0$  bosons and Higgs’ boson agreed with an experiment,
- 2° a correct value of Weinberg angle (also with radiative corrections).

It is important to mention that in order to get a correct value for Higgs’ boson mass we should consider complex Hermitian version of the theory. Thus an experiment chooses the correct version of the theory among real, Hermitian (complex Hermitian, hypercomplex Hermitian).

Ref. [14] contains all the details of these three approaches. In Ref. [14] we give also some ideas to quantize the theory using Yukawa–Efimov nonlocal quantization procedure within path integral quantization. Let us notice the following fact. One hundred years ago three men, A. Einstein, D. Hilbert and O. Klein were discussing what should be a lagrangian for a gravitational field. They decided it should be a scalar curvature for a Levi-Civita connection defined on a 4-dimensional manifold (a space-time). In that time an additional lagrangian was known. It was a lagrangian for an electromagnetic field  $-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$  coming from Maxwell equations which were known since about fifty years. Some years later T. Kaluza designed a theory where  $R_5$  (also a scalar curvature) on 5-dimensional manifold with additional symmetries is equal to  $R_4 - \frac{\lambda^2}{4}F_{\mu\nu}F^{\mu\nu}$  ( $\lambda = \frac{2\sqrt{G_N}}{c^2}$ ,  $G_N$  — a Newton constant). Now, 100 years of General Relativity, almost 150 years of Maxwell equations and almost 100 years of Kaluza’s idea we came to the conclusion that only a scalar curvature can be a lagrangian for unified field theory of all physical interactions. This will be

a scalar curvature of a connection defined on many-dimensional manifold with some symmetries and additional degrees of freedom (e.g. skew-symmetric) metric. Moreover, we should consider an action for this scalar curvature on a multi-dimensional manifold as an integral. In our approach this multi-dimensional manifold is a bundle manifold and we integrate over the bundle manifold.

Let us notice that we geometrize Higgs' fields, spontaneous symmetry breaking and Higgs' mechanism according to the Einstein program. Using our achievements from Ref. [14] we can obtain Yukawa-type terms in the lagrangian for ordinary  $\frac{1}{2}$ -spin fermions. In this way we are getting Yukawa couplings and masses for fermions from higher dimensions.

Let us give the following remark on the lagrangian in a field theory and mechanics. The lagrangian is defined as a difference between kinetic and potential energy. Let us notice the following fact. In General Relativity a scalar curvature—a lagrangian for gravitational field—is a pure kinetic energy. The same is for a lagrangian of a test particle (a lagrangian in mechanics). This is also a pure kinetic energy. Further development considered in Kaluza–Klein (Jordan–Thiry) Theory conserves this idea. In the Nonsymmetric Kaluza–Klein (Jordan–Thiry) Theory we have the same in both cases:

- 1° for a field theory (geometrized and unified),
- 2° for a mechanics (a test particle motion)—generalized Kerner–Wong–Kopczyński equation, which is a geodetic equation obtained from variational principle, where the lagrangian is “a pure kinetic energy” on multi-dimensional manifold.

There is nothing wrong in a conventional Higgs' mechanism (e.g. GSW model). Moreover, the Nonsymmetric Kaluza–Klein Theory approach gives us:

- 1° Higgs' fields as gauge fields over “a manifold of classical vacuum states” ( $S^2$  in the case of GSW model).
- 2° Proper spectrum of masses of  $W^\pm, Z_0$  and Higgs' bosons, if a geometry on  $S^2$  is complex Hermitian (really Kählerian).
- 3° Generalized Kerner–Wong–Kopczyński equation with some additional charges coupled Higgs' field to a test particle (with a possible test in an experiment).

The theory is not longer a phenomenological theory. We are getting Higgs' mechanism and masses for gauge bosons from higher dimensions.

It is a geometrical unification of gravity and electro-weak interactions as we described it above. In our meaning all fundamental interactions should be described by a multidimensional linear connection defined on a nonsymmetrically metrized fiber bundle compatible with a metric structure. A base manifold should contain a manifold of classical vacuum states. This is a bosonic part of interaction.

Fermionic sector of our approach (our own method) should be described by a fermion field defined on the mentioned manifold and coupled to a connection defined on the same manifold (a bosonic sector) in a minimal way. Our minimal coupling scheme (a covariant multidimensional coupling) has been defined in Refs [17], [18], [19], [14], [20], [21] and developed here. This minimal coupling scheme couples effectively multidimensional spinor field  $\Psi$  to a connection which is metric (with respect to a symmetric part of a multidimensional nonsymmetric metric, the notion of a nonsymmetric metric is of course an abuse of a nomination, moreover it is understable). In this

way a coupling between spinor fields ( $\frac{1}{2}$ -spin fermion fields) is consistent as in 4-dimensional case (see Refs [22], [14], [17], [18]). We consider  $\Psi$  as zero modes on a group manifold in order to avoid Planck's mass terms. We described also a minimal coupling scheme of a  $\frac{3}{2}$ -spin field for a future convenience using differential forms formalism.

For there is not any trace of supersymmetry and supergravity in an experiment we do not develop supersymmetry and supergravity approach even in  $\frac{3}{2}$ -spin fermions case. There is of course a place for such an approach in our method using supermanifolds with anticommuting coordinates (see Refs [23], [24]). Moreover, we will develop it elsewhere if the experiment gives us some important traces (future accelerators).

In the paper we consider a fermionic part of the Nonsymmetric Kaluza–Klein Theory. It means, we couple spinor fields describing fermions to a geometry on the Kaluza–Klein manifold generated by a symmetric part of a metric tensor  $\gamma_{(AB)}$  or  $\gamma_{(\tilde{A}\tilde{B})}$ . In this way we want to get a unification of gravity and Yang–Mills' fields, Higgs' fields coupled to fermions. Up to now our unification describes only a bosonic part of the theory. We work in the following way. We couple spinors to non-Abelian Kaluza–Klein Theory via new kind of gauge derivatives. These derivatives induce on many-dimensional manifold a new connection, which is metric but with a non-vanishing torsion. In this way we get a consistency between differentiation of spinors and tensors (vectors). We couple also our spinors (many-dimensional) on an extended manifold with Higgs' fields also (not only 4-dimensional Yang–Mills' fields) and we give a comprehensive treatment of a spinor coupling to a bosonic part of GSW (Glashow–Salam–Weinberg) model in our approach. We get a unification of Yukawa coupling e.g. in GSW model. We suppose that our spinor fields are zero-

modes of  $\overset{\text{(int)}}{\mathbb{D}}$  operator. In this way we remove very heavy fermions (with masses of the Planck's mass order) from the theory. We develop these fermion fields in zero-modes functions and also in generalized harmonics on a compact manifold  $M = G/G_0$ . In the case of GSW-model we have  $S^2$  and of course spherical harmonics. We get several interesting terms in the lagrangian for fermions and we interpret them. We get several terms coming to mass terms and also to some mixings. Eventually we proceed a symmetry breaking from  $SO(1, n+3)$  ( $Spin(1, n+3)$ ),  $SO(1, n+n_1+3)$  ( $Spin(1, n+n_1+3)$ ) or  $SO(1, 19)$  ( $Spin(1, 19)$ ) to a Lorentz group  $SO(1, 3)$  ( $SL(2, \mathbb{C})$ ) getting a tower of spinor fields (0-forms and 1-forms). Spinors (zero modes) are coupled by new gauge derivatives. This is a new point.

Let us give some remarks on a coupling between Rarita–Schwinger field ( $\frac{3}{2}$ -spin fermion field) and gravity. For a very long time a coupling between gravity and  $\frac{3}{2}$ -spin fields was badly defined. In [25], [26] a new coupling was introduced between gravity and Rarita–Schwinger fields, which solved the problem of coupling to gravity. It was obtained by two independent ways. In supergravitation theory [25], and by using the formalism of differential forms [26]. We are using the second approach and extending it to the Kaluza–Klein theory. The minimal coupling for Rarita–Schwinger fields to an electromagnetic field (in general, to any gauge field) is inconsistent for we have the Velo–Zwanziger paradox [27] in the scheme. The Rarita–Schwinger equation is relativistically covariant but solutions can be acausal. This happens due to algebraic constraints which are differential consequences of the Rarita–Schwinger equation [27], [28]. These constraints depend on the strength of the electromagnetic field. In the paper [29] we consider new kinds of gauge derivatives for spin- $\frac{3}{2}$  fields (see [17], [18]). In such a way we generalize the minimal coupling scheme. We use differential forms for Rarita–Schwinger fields as in [26]. But we define

spin- $\frac{3}{2}$  fields on the Kaluza–Klein manifold. This is of course a 5-dimensional case. From this we get a new term in the lagrangian. This term describes the interaction between the electromagnetic field and a dipole electric moment of the  $\frac{3}{2}$ -spinor field of value about  $10^{-31}$  [cm]  $q$  (see Ref. [29]). Thus this term is very small. The term is an “interference” effect between the gravitational and electromagnetic fields. It breaks PC and T and is similar to a term given in Refs [17], [18], [20]. Our new term has an important consequence. Namely, it implies that the first differential consequences for the Rarita–Schwinger equation (obtained from this lagrangian) are differential equations. We do not obtain algebraic constraints involving electromagnetic field. Thus the Velo–Zwanziger paradox is avoided. In Ref. [30] one gets also a nonminimal coupling between the Rarita–Schwinger field and an electromagnetic field. The additional term in Ref. [30] does not violate the PC symmetry and is much different from our term. The most fundamental difference between  $N = 2$  supergravity approach (see Ref. [30]) and our approach (see Ref. [29]) consists not only in the PC breaking, but in the fact that our Rarita–Schwinger field is *charged* and Rarita–Schwinger field in Ref. [30] is *uncharged*. In this way we have the ordinary coupling between the electromagnetic field and the Rarita–Schwinger field plus new term, and we avoid the Velo–Zwanziger paradox. In [30] this does not occur, because Rarita–Schwinger field is *uncharged* and the paradox does not take place. However, we pay a price for avoiding the Velo–Zwanziger paradox. Our theory breaks PC. Fortunately this breaking is very small. We point out that our theory is not a supergravity-like theory, and we do not use anticommuting Majorana spinors.

Afterwards we consequently develop the formalism in the non-Abelian Kaluza–Klein Theory. The coupling between  $\frac{3}{2}$ -spin field is consistently using differential forms formalism (see Ref. [20]). This coupling (in Ref. [20]) is a generalization of a minimal coupling scheme and is going to some kind of PC-breaking terms (of the same order as in 5-dimensional case). In the paper we extend the approach to the case with spontaneous symmetry breaking and Higgs’ mechanism. Moreover, the paper is devoted basically to  $\frac{1}{2}$ -spin fermion fields. All the fields considered here are zero modes of the group manifold, which is different from previous approaches (see Refs [20], [31], [32]). This is supposed in order to avoid Planck’s mass terms in the lagrangian for fermion fields. A detailed formalism for  $\frac{3}{2}$ -spin fermion fields will be developed elsewhere as we mention in the text. We expect to avoid the Velo–Zwanziger paradox also in the case with spontaneous symmetry breaking and Higgs’ mechanism, getting also Yukawa-type terms. The absence of Velo–Zwanziger paradox is quite obvious because differential consequences of the field equation for  $\frac{3}{2}$ -spin fermion field are here *differential equations, not algebraic constraints*.

The paper is divided into five sections. In the second section we describe the Nonsymmetric Kaluza–Klein Theory in general non-Abelian case and also with a spontaneous symmetry breaking taken into account. In the third section we describe a bosonic part of GSW-model in the theory. In the fourth section we deal with spinor fields on a manifold  $P$ . The fifth section is devoted to the lagrangian of fermions in the theory. We consider and discuss several possibilities of such a lagrangian. We find several interesting terms in lagrangians. We develop (as we mentioned before) spinor fields into a generalized Fourier series of generalized harmonics and zero-modes function of  $\mathcal{D}^{(\text{int})}$ . We consider also the problem of chiral fermions using arguments of Atiyah–Singer index theorem. Eventually we proceed a symmetry breaking to the Lorentz group getting a tower of spinor fields. This is a new point in the Nonsymmetric Kaluza–Klein Theory.

In Appendix A we give some elements of Clifford algebras and Dirac matrices applicable for our theory. In Appendix B we consider covariant derivatives in our theory. Appendix C is devoted

to Atiyah–Singer index theorem applicable for  $\overset{(\text{int})}{\mathbb{D}}$  elliptic operator defined on a compact group  $G$  or  $H$ . In Conclusions we give some prospects for further research.

## 2 Elements of the Nonsymmetric Kaluza–Klein Theory in general non-Abelian case and with spontaneous symmetry breaking and Higgs’ mechanism

Let  $\underline{P}$  be a principal fiber bundle over a space-time  $E$  with a structural group  $G$  which is a semisimple Lie group. On a space-time  $E$  we define a nonsymmetric tensor  $g_{\mu\nu} = g_{(\mu\nu)} + g_{[\mu\nu]}$  such that

$$\begin{aligned} g &= \det(g_{\mu\nu}) \neq 0 \\ \tilde{g} &= \det(g_{(\mu\nu)}) \neq 0. \end{aligned} \quad (2.1)$$

$g_{[\mu\nu]}$  is called as usual a skewon field (e.g. in NGT, see Refs [6, 9]). We define on  $E$  a nonsymmetric connection compatible with  $g_{\mu\nu}$  such that

$$\overline{D}g_{\alpha\beta} = g_{\alpha\delta}\overline{Q}^\delta_{\beta\gamma}(\overline{\Gamma})\overline{\theta}^\gamma \quad (2.2)$$

where  $\overline{D}$  is an exterior covariant derivative for a connection  $\overline{\omega}^\alpha_\beta = \overline{\Gamma}^\alpha_{\beta\gamma}\overline{\theta}^\gamma$  and  $\overline{Q}^\alpha_{\beta\delta}$  is its torsion. We suppose also

$$\overline{Q}^\alpha_{\beta\alpha}(\overline{\Gamma}) = 0. \quad (2.3)$$

We introduce on  $E$  a second connection

$$\overline{W}^\alpha_\beta = \overline{W}^\alpha_{\beta\gamma}\overline{\theta}^\gamma \quad (2.4)$$

such that

$$\overline{W}^\alpha_\beta = \overline{\omega}^\alpha_\beta - \frac{2}{3}\delta^\alpha_\beta\overline{W} \quad (2.5)$$

$$\overline{W} = \overline{W}_\gamma\overline{\theta}^\gamma = \frac{1}{2}(\overline{W}^\sigma_{\gamma\sigma} - \overline{W}^\sigma_{\sigma\gamma})\overline{\theta}^\gamma. \quad (2.6)$$

Now we turn to nonsymmetric metrization of a bundle  $\underline{P}$ . We define a nonsymmetric tensor  $\gamma$  on a bundle manifold  $P$  such that

$$\gamma = \pi^*g \oplus \ell_{ab}\theta^a \otimes \theta^b \quad (2.7)$$

where  $\pi$  is a projection from  $P$  to  $E$ . On  $\underline{P}$  we define a connection  $\omega$  (a 1-form with values in a Lie algebra  $\mathfrak{g}$  of  $G$ ). In this way we can introduce on  $P$  (a bundle manifold) a frame  $\theta^A = (\pi^*(\overline{\theta}^\alpha), \theta^a)$  such that

$$\theta^a = \lambda\omega^a, \quad \omega = \omega^a X_a, \quad a = 5, 6, \dots, n+4, \quad n = \dim G = \dim \mathfrak{g}, \quad \lambda = \text{const.}$$

Thus our nonsymmetric tensor looks like

$$\gamma = \gamma_{AB}\theta^A \otimes \theta^B, \quad A, B = 1, 2, \dots, n+4, \quad (2.8)$$

$$\ell_{ab} = h_{ab} + \mu k_{ab}, \quad (2.9)$$

where  $h_{ab}$  is a biinvariant Killing–Cartan tensor on  $G$  and  $k_{ab}$  is a right-invariant skew-symmetric tensor on  $G$ ,  $\mu = \text{const.}$

We have

$$\begin{aligned} h_{ab} &= C_{ad}^c C_{bc}^d = h_{ab} \\ k_{ab} &= -k_{ba} \end{aligned} \quad (2.10)$$

Thus we can write

$$\overline{\gamma}(X, Y) = \overline{g}(\pi' X, \pi' Y) + \lambda^2 h(\omega(X), \omega(Y)) \quad (2.11)$$

$$\underline{\gamma}(X, Y) = \underline{g}(\pi' X, \pi' Y) + \lambda^2 k(\omega(X), \omega(Y)) \quad (2.12)$$

( $C_{bc}^a$  are structural constants of the Lie algebra  $\mathfrak{g}$ ).

$\overline{\gamma}$  is the symmetric part of  $\gamma$  and  $\underline{\gamma}$  is the antisymmetric part of  $\gamma$ . We have as usual

$$[X_a, X_b] = C_{ab}^c X_c \quad (2.13)$$

and

$$\Omega = \frac{1}{2} H_{\mu\nu}^a \theta^\mu \wedge \theta^\nu \quad (2.14)$$

is a curvature of the connection  $\omega$ ,

$$\Omega = d\omega + \frac{1}{2}[\omega, \omega]. \quad (2.15)$$

The frame  $\theta^A$  on  $P$  is partially nonholonomic. We have

$$d\theta^a = \frac{\lambda}{2} \left( H_{\mu\nu}^a \theta^\mu \wedge \theta^\nu - \frac{1}{\lambda^2} C_{bc}^a \theta^b \wedge \theta^c \right) \neq 0 \quad (2.16)$$

even if the bundle  $\underline{P}$  is trivial, i.e. for  $\Omega = 0$ . This is different than in an electromagnetic case (see Ref. [3]). Our nonsymmetric metrization of a principal fiber bundle gives us a right-invariant structure on  $P$  with respect to an action of a group  $G$  on  $P$  (see Ref. [3] for more details). Having  $P$  nonsymmetrically metrized one defines two connections on  $P$  right-invariant with respect to an action of a group  $G$  on  $P$ . We have

$$\gamma_{AB} = \left( \begin{array}{c|c} g_{\alpha\beta} & 0 \\ \hline 0 & \ell_{ab} \end{array} \right) \quad (2.17)$$

in our left horizontal frame  $\theta^A$ .

$$D\gamma_{AB} = \gamma_{AD} Q_{BC}^D(\Gamma) \theta^C \quad (2.18)$$

$$Q_{BD}^D(\Gamma) = 0 \quad (2.19)$$

where  $D$  is an exterior covariant derivative with respect to a connection  $\omega_{AB}^A = \Gamma_{BC}^A \theta^C$  on  $P$  and  $Q_{BC}^A(\Gamma)$  its torsion. One can solve Eqs (2.18)–(2.19) getting the following results

$$\omega_{AB}^A = \left( \begin{array}{c|c} \pi^*(\overline{\omega}_{\alpha\beta}^{\alpha}) - \ell_{db} g^{\mu\alpha} L_{\mu\beta}^d \theta^b & L_{\beta\gamma}^a \theta^\gamma \\ \hline \ell_{bd} g^{\alpha\beta} (2H_{\gamma\beta}^d - L_{\gamma\beta}^d \theta^\gamma) & \tilde{\omega}_b^a \end{array} \right) \quad (2.20)$$

where  $g^{\mu\alpha}$  is an inverse tensor of  $g_{\alpha\beta}$

$$g_{\alpha\beta} g^{\gamma\beta} = g_{\beta\alpha} g^{\beta\gamma} = \delta_{\alpha}^{\gamma}, \quad (2.21)$$



$L^d_{\gamma\beta} = -L^a_{\beta\gamma}$  is an Ad-type tensor on  $P$  such that

$$\ell_{dc}g_{\mu\beta}g^{\gamma\mu}L^d_{\gamma\alpha} + \ell_{cd}g_{\alpha\mu}g^{\mu\gamma}L^d_{\beta\gamma} = 2\ell_{cd}g_{\alpha\mu}g^{\mu\gamma}H^d_{\beta\gamma}, \quad (2.22)$$

$\tilde{\omega}^a_b = \tilde{\Gamma}^a_{bc}\theta^c$  is a connection on an internal space (typical fiber) compatible with a metric  $\ell_{ab}$  such that

$$\ell_{db}\tilde{\Gamma}^d_{ac} + \ell_{ad}\tilde{\Gamma}^d_{cb} = -\ell_{ab}C^d_{ac} \quad (2.23)$$

$$\tilde{\Gamma}^a_{ba} = 0, \quad \tilde{\Gamma}^a_{bc} = -\tilde{\Gamma}^a_{cb} \quad (2.24)$$

and of course  $\tilde{Q}^a_{ba}(\tilde{\Gamma}) = 0$  where  $\tilde{Q}^a_{bc}(\Gamma)$  is a torsion of the connection  $\tilde{\omega}^a_b$ .

We also introduce an inverse tensor of  $g_{(\alpha\beta)}$

$$g_{(\alpha\beta)}\tilde{g}^{(\alpha\gamma)} = \delta^\gamma_\beta. \quad (2.25)$$

We introduce a second connection on  $P$  defined as

$$W^A_B = \omega^A_B - \frac{4}{3(n+2)}\delta^A_B\overline{W}. \quad (2.26)$$

$\overline{W}$  is a horizontal one form

$$\overline{W} = \text{hor } \overline{W} \quad (2.27)$$

$$\overline{W} = \overline{W}_\nu\theta^\nu = \frac{1}{2}(\overline{W}^\sigma_{\nu\sigma} - \overline{W}^\sigma_{\sigma\nu}). \quad (2.28)$$

In this way we define on  $P$  all analogues of four-dimensional quantities from NGT (see Refs [6, 33, 34, 35]). It means,  $(n+4)$ -dimensional analogues from Moffat theory of gravitation, i.e. two connections and a nonsymmetric metric  $\gamma_{AB}$ . Those quantities are right-invariant with respect to an action of a group  $G$  on  $P$ . One can calculate a scalar curvature of a connection  $W^A_B$  getting the following result (see Refs [1, 3]):

$$R(W) = \overline{R}(\overline{W}) - \frac{\lambda^2}{4}(2\ell_{cd}H^cH^d - \ell_{cd}L^{c\mu\nu}H^d_{\mu\nu}) + \tilde{R}(\tilde{\Gamma}) \quad (2.29)$$

where

$$R(W) = \gamma^{AB}(R^C_{ABC}(W) + \frac{1}{2}R^C_{CAB}(W)) \quad (2.30)$$

is a Moffat–Ricci curvature scalar for the connection  $W^A_B$ ,  $\overline{R}(\overline{W})$  is a Moffat–Ricci curvature scalar for the connection  $\overline{W}^\alpha_\beta$ , and  $\tilde{R}(\tilde{\Gamma})$  is a Moffat–Ricci curvature scalar for the connection  $\tilde{\omega}^a_b$ ,

$$H^a = g^{[\mu\nu]}H^a_{\mu\nu} \quad (2.31)$$

$$L^{a\mu\nu} = g^{\alpha\mu}g^{\beta\nu}L^a_{\alpha\beta}. \quad (2.32)$$

Usually in ordinary (symmetric) Kaluza–Klein Theory one has  $\lambda = 2\frac{\sqrt{G_N}}{c^2}$ , where  $G_N$  is a Newtonian gravitational constant and  $c$  is the speed of light. In our system of units  $G_N = c = 1$  and  $\lambda = 2$ . This is the same as in Nonsymmetric Kaluza–Klein Theory in an electromagnetic case (see Refs [4, 9]). In the non-Abelian Kaluza–Klein Theory which unifies GR and Yang–Mills field theory we have a Yang–Mills lagrangian and a cosmological term. Here we have

$$\mathcal{L}_{\text{YM}} = -\frac{1}{8\pi}\ell_{cd}(2H^cH^d - L^{c\mu\nu}H^d_{\mu\nu}) \quad (2.33)$$

and  $\tilde{R}(\tilde{\Gamma})$  plays a role of a cosmological term.

In order to incorporate a spontaneous symmetry breaking and Higgs' mechanism in our geometrical unification of gravitation and Yang–Mills' fields we consider a fiber bundle  $\underline{P}$  over a base manifold  $E \times G/G_0$ , where  $E$  is a space-time,  $G_0 \subset G$ ,  $G_0, G$  are semisimple Lie groups. Thus we are going to combine a Kaluza–Klein theory with a dimensional reduction procedure.

Let  $\underline{P}$  be a principal fiber bundle over  $V = E \times M$  with a structural group  $H$  and with a projection  $\pi$ , where  $M = G/G_0$  is a homogeneous space,  $G$  is a semisimple Lie group and  $G_0$  its semisimple Lie subgroup. Let us suppose that  $(V, \gamma)$  is a manifold with a nonsymmetric metric tensor

$$\gamma_{AB} = \gamma_{(AB)} + \gamma_{[AB]}. \quad (2.34)$$

The signature of the tensor  $\gamma$  is  $(+---, \underbrace{---\dots-}_{n_1})$ . Let us introduce a natural frame on  $\underline{P}$

$$\theta^{\tilde{A}} = (\pi^*(\theta^A), \theta^0 = \lambda\omega^a), \quad \lambda = \text{const}. \quad (2.35)$$

It is convenient to introduce the following notation. Capital Latin indices with tilde  $\tilde{A}, \tilde{B}, \tilde{C}$  run  $1, 2, 3, \dots, m+4$ ,  $m = \dim H + \dim M = n + \dim M = n + n_1$ ,  $n_1 = \dim M$ ,  $n = \dim H$ . Lower Greek indices  $\alpha, \beta, \gamma, \delta = 1, 2, 3, 4$  and lower Latin indices  $a, b, c, d = n_1 + 5, n_2 + 5, \dots, n_1 + 6, \dots, m+4$ . Capital Latin indices  $A, B, C = 1, 2, \dots, n_1 + 4$ . Lower Latin indices with tilde  $\tilde{a}, \tilde{b}, \tilde{c}$  run  $5, 6, \dots, n_1 + 4$ . The symbol “ $\tilde{\phantom{x}}$ ” over  $\theta^A$  and other quantities indicates that these quantities are defined on  $V$ . We have of course

$$n_1 = \dim G - \dim G_0 = n_2 - (n_2 - n_1),$$

where  $\dim G = n_2$ ,  $\dim G_0 = n_2 - n_1$ ,  $m = n_1 + n$ .

On the group  $H$  we define a bi-invariant (symmetric) Killing–Cartan tensor

$$h(A, B) = h_{ab}A^aB^b. \quad (2.36)$$

We suppose  $H$  is semisimple, it means  $\det(h_{ab}) \neq 0$ . We define a skew-symmetric right-invariant tensor on  $H$

$$k(A, B) = k_{bc}A^bB^c, \quad k_{bc} = -k_{cb}.$$

Let us turn to the nonsymmetric metrization of  $\underline{P}$ .

$$\kappa(X, Y) = \gamma(X, Y) + \lambda^2 \ell_{ab} \omega^a(X) \omega^b(Y) \quad (2.37)$$

where

$$\ell_{ab} = h_{ab} + \xi k_{ab} \quad (2.38)$$

is a nonsymmetric right-invariant tensor on  $H$ . One gets in a matrix form (in the natural frame (2.35))

$$\kappa_{\tilde{A}\tilde{B}} = \left( \begin{array}{c|c} \gamma_{AB} & 0 \\ \hline 0 & \ell_{ab} \end{array} \right), \quad (2.39)$$

$\det(\ell_{ab}) \neq 0$ ,  $\xi = \text{const}$  and real, then

$$\ell_{ab} \ell^{ac} = \ell_{ba} \ell^{ca} = \delta^c_b. \quad (2.40)$$

The signature of the tensor  $\kappa$  is  $(+, ---, \underbrace{- \cdots -}_{n_1}, \underbrace{- \cdots -}_n)$ . As usual, we have commutation relations for Lie algebra of  $H$ ,  $\mathfrak{h}$

$$[X_a, X_b] = C_{ab}^c X_c. \quad (2.41)$$

This metrization of  $\underline{P}$  is right-invariant with respect to an action of  $H$  on  $P$ .

Now we should nonsymmetrically metrize  $M = G/G_0$ .  $M$  is a homogeneous space for  $G$  (with left action of group  $G$ ). Let us suppose that the Lie algebra of  $G$ ,  $\mathfrak{g}$  has the following reductive decomposition

$$\mathfrak{g} = \mathfrak{g}_0 \dot{+} \mathfrak{m} \quad (2.42)$$

where  $\mathfrak{g}_0$  is a Lie algebra of  $G_0$  (a subalgebra of  $\mathfrak{g}$ ) and  $\mathfrak{m}$  (the complement to the subalgebra  $\mathfrak{g}_0$ ) is  $\text{Ad } G_0$  invariant,  $\dot{+}$  means a direct sum. Such a decomposition might be not unique, but we assume that one has been chosen. Sometimes one assumes a stronger condition for  $\mathfrak{m}$ , the so called symmetry requirement,

$$[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{g}_0. \quad (2.43)$$

Let us introduce the following notation for generators of  $\mathfrak{g}$ :

$$Y_i \in \mathfrak{g}, \quad Y_{\bar{i}} \in \mathfrak{m}, \quad Y_{\bar{a}} \in \mathfrak{g}_0. \quad (2.44)$$

This is a decomposition of a basis of  $\mathfrak{g}$  according to (2.42). We define a symmetric metric on  $M$  using a Killing–Cartan form on  $G$  in a classical way. We call this tensor  $h_0$ .

Let us define a tensor field  $h^0(x)$  on  $G/G_0$ ,  $x \in G/G_0$ , using tensor field  $h$  on  $G$ . Moreover, if we suppose that  $h$  is a biinvariant metric on  $G$  (a Killing–Cartan tensor) we have a simpler construction.

The complement  $\mathfrak{m}$  is a tangent space to the point  $\{\varepsilon G_0\}$  of  $M$ ,  $\varepsilon$  is a unit element of  $G$ . We restrict  $h$  to the space  $\mathfrak{m}$  only. Thus we have  $h^0(\{\varepsilon G_0\})$  at one point of  $M$ . Now we propagate  $h^0(\{fG_0\})$  using a left action of the group  $G$

$$h^0(\{fG_0\}) = (L_f^{-1})^*(h^0(\{\varepsilon G_0\})).$$

$h^0(\{\varepsilon G_0\})$  is of course  $\text{Ad } G_0$  invariant tensor defined on  $\mathfrak{m}$  and  $L_f^* h^0 = h^0$ .

We define on  $M$  a skew-symmetric 2-form  $k^0$ . Now we introduce a natural frame on  $M$ . Let  $f^i_{jk}$  be structure constants of the Lie algebra  $\mathfrak{g}$ , i.e.

$$[Y_j, Y_k] = f^i_{jk} Y_i. \quad (2.45)$$

$Y_j$  are generators of the Lie algebra  $\mathfrak{g}$ . Let us take a local section  $\sigma : V \rightarrow G/G_0$  of a natural bundle  $G \mapsto G/G_0$  where  $V \subset M = G/G_0$ . The local section  $\sigma$  can be considered as an introduction of a coordinate system on  $M$ .

Let  $\omega_{MC}$  be a left-invariant Maurer–Cartan form and let

$$\omega_{MC}^\sigma = \sigma^* \omega_{MC}. \quad (2.46)$$

Using decomposition (2.42) we have

$$\omega_{MC}^\sigma = \omega_{\sigma_0}^\sigma + \omega_{\mathfrak{m}}^\sigma = \widehat{\theta^i} Y_{\bar{i}} + \bar{t}^{\bar{a}} Y_{\bar{a}}. \quad (2.47)$$

It is easy to see that  $\bar{\theta}^{\tilde{a}}$  is the natural (left-invariant) frame on  $M$  and we have

$$h^0 = h^0_{\tilde{a}\tilde{b}} \bar{\theta}^{\tilde{a}} \otimes \bar{\theta}^{\tilde{b}} \quad (2.48)$$

$$k^0 = k^0_{\tilde{a}\tilde{b}} \bar{\theta}^{\tilde{a}} \wedge \bar{\theta}^{\tilde{b}}. \quad (2.49)$$

According to our notation  $\tilde{a}, \tilde{b} = 5, 6, \dots, n_1 + 4$ .

Thus we have a nonsymmetric metric on  $M$

$$\gamma_{\tilde{a}\tilde{b}} = r^2 (h^0_{\tilde{a}\tilde{b}} + \zeta k^0_{\tilde{a}\tilde{b}}) = r^2 g_{\tilde{a}\tilde{b}}. \quad (2.50)$$

Thus we are able to write down the nonsymmetric metric on  $V = E \times M = E \times G/G_0$

$$\gamma_{AB} = \left( \begin{array}{c|c} g_{\alpha\beta} & 0 \\ \hline 0 & r^2 g_{\tilde{a}\tilde{b}} \end{array} \right) \quad (2.51)$$

where

$$\begin{aligned} g_{\alpha\beta} &= g_{(\alpha\beta)} + g_{[\alpha\beta]} \\ g_{\tilde{a}\tilde{b}} &= h^0_{\tilde{a}\tilde{b}} + \zeta k^0_{\tilde{a}\tilde{b}} \\ k^0_{\tilde{a}\tilde{b}} &= -k^0_{\tilde{b}\tilde{a}} \\ h^0_{\tilde{a}\tilde{b}} &= h^0_{\tilde{b}\tilde{a}}, \end{aligned}$$

$\alpha, \beta = 1, 2, 3, 4$ ,  $\tilde{a}, \tilde{b} = 5, 6, \dots, n_1 + 4 = \dim M + 4 = \dim G - \dim G_0 + 4$ . The frame  $\bar{\theta}^{\tilde{a}}$  is unholonomic:

$$d\bar{\theta}^{\tilde{a}} = \frac{1}{2} \kappa^{\tilde{a}}_{\tilde{b}\tilde{c}} \bar{\theta}^{\tilde{b}} \wedge \bar{\theta}^{\tilde{c}} \quad (2.52)$$

where  $\kappa^{\tilde{a}}_{\tilde{b}\tilde{c}}$  are coefficients of nonholonomicity and depend on the point of the manifold  $M = G/G_0$  (they are not constant in general). They depend on the section  $\sigma$  and on the constants  $f^{\tilde{a}}_{\tilde{b}\tilde{c}}$ .

We have here three groups  $H, G, G_0$ . Let us suppose that there exists a homomorphism  $\mu$  between  $G_0$  and  $H$ ,

$$\mu : G_0 \rightarrow H \quad (2.53)$$

such that a centralizer of  $\mu(G_0)$  in  $H$ ,  $C^\mu$  is isomorphic to  $G$ .  $C^\mu$ , a centralizer of  $\mu(G_0)$  in  $H$ , is a set of all elements of  $H$  which commute with elements of  $\mu(G_0)$ , which is a subgroup of  $H$ . This means that  $H$  has the following structure,  $C^\mu = G$ .

$$\mu(G_0) \otimes G \subset H. \quad (2.54)$$

If  $\mu$  is a isomorphism between  $G_0$  and  $\mu(G_0)$  one gets

$$G_0 \otimes G \subset H. \quad (2.55)$$

Let us denote by  $\mu'$  a tangent map to  $\mu$  at a unit element. Thus  $\mu'$  is a differential of  $\mu$  acting on the Lie algebra elements. Let us suppose that the connection  $\omega$  on the fiber bundle  $\underline{P}$  is invariant under group action of  $G$  on the manifold  $V = E \times G/G_0$ . According to Refs [36, 37, 38, 39] this means the following.

Let  $e$  be a local section of  $\underline{P}$ ,  $e : V \subset U \rightarrow P$  and  $A = e^* \omega$ . Then for every  $g \in G$  there exists a gauge transformation  $\rho_g$  such that

$$f^*(g)A = \text{Ad}_{\rho_g^{-1}} A + \rho_g^{-1} dg, \quad (2.56)$$

$f^*$  means a pull-back of the action  $f$  of the group  $G$  on the manifold  $V$ . According to Refs [37, 38, 39, 40, 41, 42] (see also Refs [43, 44, 45]) we are able to write a general form for such an  $\omega$ . Following Ref. [39] we have

$$\omega = \tilde{\omega}_E + \mu' \circ \omega^\sigma_0 + \Phi \circ \omega^\sigma_{\mathfrak{m}}. \quad (2.57)$$

(An action of a group  $G$  on  $V = E \times G/G_0$  means left multiplication on a homogeneous space  $M = G/G_0$ .) where  $\omega^\sigma_0 + \omega^\sigma_{\mathfrak{m}} = \omega^\sigma_{MC}$  are components of the pull-back of the Maurer–Cartan form from the decomposition (2.47),  $\tilde{\omega}_E$  is a connection defined on a fiber bundle  $Q$  over a space-time  $E$  with structural group  $C^\mu$  and a projection  $\pi_E$ . Moreover,  $C^\mu = G$  and  $\tilde{\omega}_E$  is a 1-form with values in the Lie algebra  $\mathfrak{g}$ . This connection describes an ordinary Yang–Mills’ field gauge group  $G = C^\mu$  on the space-time  $E$ .  $\Phi$  is a function on  $E$  with values in the space  $\tilde{S}$  of linear maps

$$\Phi : \mathfrak{m} \rightarrow \mathfrak{h} \quad (2.58)$$

satisfying

$$\Phi([X_0, X]) = [\mu' X_0, \Phi(X)], \quad X_0 \in \mathfrak{g}_0. \quad (2.59)$$

Thus

$$\begin{aligned} \tilde{\omega}_E &= \tilde{\omega}_E^i Y_i, \quad Y_i \in \mathfrak{g}, \\ \omega^\sigma_0 &= \hat{\theta}^i Y_i, \quad Y_i \in \mathfrak{g}_0, \\ \omega^\sigma_{\mathfrak{m}} &= \bar{\theta}^{\bar{a}} Y_{\bar{a}}, \quad Y_{\bar{a}} \in \mathfrak{m}. \end{aligned} \quad (2.60)$$

Let us write condition (2.57) in the base of left-invariant form  $\hat{\theta}^i, \bar{\theta}^{\bar{a}}$ , which span respectively dual spaces to  $\mathfrak{g}_0$  and  $\mathfrak{m}$  (see Refs [46, 47]). It is easy to see that

$$\Phi \circ \omega^\sigma_{\mathfrak{m}} = \Phi_{\bar{a}}^a(x) \bar{\theta}^{\bar{a}} X_a, \quad X_a \in \mathfrak{h} \quad (2.61)$$

and

$$\mu' = \mu_{\hat{i}}^a \hat{\theta}^i X_a. \quad (2.62)$$

From (2.59) one gets

$$\Phi_b^c(x) f_{i\bar{a}}^{\bar{b}} = \mu_{\hat{i}}^a \Phi_{\bar{a}}^b(x) C_{ab}^c \quad (2.63)$$

where  $f_{i\bar{a}}^{\bar{b}}$  are structure constants of the Lie algebra  $\mathfrak{g}$  and  $C_{ab}^c$  are structure constants of the Lie algebra  $\mathfrak{h}$ . Eq. (2.63) is a constraint on the scalar field  $\Phi_{\bar{a}}^a(x)$ . For a curvature of  $\omega$  one gets

$$\begin{aligned} \Omega &= \frac{1}{2} H^C_{AB} \theta^A \wedge \theta^B X_C = \frac{1}{2} \tilde{H}^i_{\mu\nu} \theta^\mu \wedge \theta^\nu \alpha_i^c X_c + \overset{\text{gauge}}{\nabla_\mu} \Phi_{\bar{a}}^c \theta^\mu \wedge \theta^{\bar{a}} X_c \\ &\quad + \frac{1}{2} C_{ab}^c \Phi_{\bar{a}}^a \Phi_{\bar{b}}^b \theta^{\bar{a}} \wedge \theta^{\bar{b}} X_c - \frac{1}{2} \Phi_{\bar{a}}^c f_{\bar{a}\bar{b}}^{\bar{d}} \theta^{\bar{a}} \wedge \theta^{\bar{b}} X_c - \mu_{\hat{i}}^c f_{\bar{a}\bar{b}}^{\bar{d}} \theta^{\bar{a}} \wedge \theta^{\bar{b}} X_c. \end{aligned} \quad (2.64)$$

Thus we have

$$H^c_{\mu\nu} = \alpha^c_i \tilde{H}^i_{\mu\nu} \quad (2.65)$$

$$H^c_{\mu\tilde{a}} = \overset{\text{gauge}}{\nabla}_\mu \Phi^c_{\tilde{a}} = -H^c_{\tilde{a}\mu} \quad (2.66)$$

$$H^c_{\tilde{a}\tilde{b}} = C^c_{ab} \cdot \Phi^a_{\tilde{a}} \Phi^b_{\tilde{b}} - \mu^c_i \hat{f}^i_{\tilde{a}\tilde{b}} - \Phi^c_{\tilde{a}} \hat{f}^{\tilde{d}}_{\tilde{a}\tilde{b}} \quad (2.67)$$

where  $\overset{\text{gauge}}{\nabla}_\mu$  means gauge derivative with respect to the connection  $\tilde{\omega}_E$  defined on a bundle  $Q$  over a space-time  $E$  with a structural group  $G$

$$Y_i = \alpha^c_i X_c. \quad (2.68)$$

$\tilde{H}^i_{\mu\nu}$  is the curvature of the connection  $\tilde{\omega}_E$  in the base  $\{Y_i\}$ , generators of the Lie algebra of the Lie group  $G$ ,  $\mathfrak{g}$ ,  $\alpha^c_i$  is the matrix which connects  $\{Y_i\}$  with  $\{X_c\}$ . Now we would like to remind that indices  $a, b, c$  refer to the Lie algebra  $\mathfrak{h}$ ,  $\tilde{a}, \tilde{b}, \tilde{c}$  to the space  $\mathfrak{m}$  (tangent space to  $M$ ),  $\hat{i}, \hat{j}, \hat{k}$  to the Lie algebra  $\mathfrak{g}_0$  and  $i, j, k$  to the Lie algebra of the group  $G$ ,  $\mathfrak{g}$ . The matrix  $\alpha^c_i$  establishes a direct relation between generators of the Lie algebra of the subgroup of the group  $H$  isomorphic to the group  $G$ .

Let us come back to a construction of the Nonsymmetric Kaluza–Klein Theory on a manifold  $P$ . We should define connections. First of all, we should define a connection compatible with a nonsymmetric tensor  $\gamma_{AB}$ , Eq. (2.51),

$$\tilde{\omega}^A_B = \bar{\Gamma}^A_{BC} \theta^C \quad (2.69)$$

$$\bar{D}\gamma_{AB} = \gamma_{AD} \bar{Q}^D_{BC}(\bar{\Gamma}) \theta^C \quad (2.70)$$

$$\bar{Q}^D_{BD}(\bar{\Gamma}) = 0$$

where  $\bar{D}$  is the exterior covariant derivative with respect to  $\tilde{\omega}^A_B$  and  $\bar{Q}^D_{BC}(\bar{\Gamma})$  its torsion.

Using (2.51) one easily finds that the connection (2.69) has the following shape

$$\tilde{\omega}^A_B = \left( \begin{array}{c|c} \pi^*_E(\tilde{\omega}^\alpha_\beta) & 0 \\ \hline 0 & \hat{\tilde{\omega}}^{\tilde{a}}_{\tilde{b}} \end{array} \right) \quad (2.71)$$

where  $\tilde{\omega}^\alpha_\beta = \bar{\Gamma}^\alpha_{\beta\gamma} \bar{\theta}^\gamma$  is a connection on the space-time  $E$  and  $\hat{\tilde{\omega}}^{\tilde{a}}_{\tilde{b}} = \hat{\Gamma}^{\tilde{a}}_{\tilde{b}\tilde{c}} \bar{\theta}^{\tilde{c}}$  on the manifold  $M = G/G_0$  with the following properties

$$\bar{D}g_{\alpha\beta} = g_{\alpha\delta} \bar{Q}^\delta_{\beta\gamma}(\bar{\Gamma}) \bar{\theta}^\gamma = 0 \quad (2.72)$$

$$\bar{Q}^\alpha_{\beta\alpha}(\bar{\Gamma}) = 0 \quad (2.73)$$

$$\hat{\bar{D}}g_{\tilde{a}\tilde{b}} = g_{\tilde{a}\tilde{d}} \hat{\bar{Q}}^{\tilde{d}}_{\tilde{b}\tilde{c}}(\hat{\bar{\Gamma}}). \quad (2.74)$$

$$\hat{\bar{Q}}^{\tilde{d}}_{\tilde{b}\tilde{d}}(\hat{\bar{\Gamma}}) = 0$$

$\bar{D}$  is an exterior covariant derivative with respect to a connection  $\tilde{\omega}^\alpha_\beta$ .  $\bar{Q}^\alpha_{\beta\gamma}$  is a tensor of torsion of a connection  $\tilde{\omega}^\alpha_\beta$ .  $\hat{\bar{D}}$  is an exterior covariant derivative of a connection  $\hat{\tilde{\omega}}^{\tilde{a}}_{\tilde{b}}$  and  $\hat{\bar{Q}}^{\tilde{a}}_{\tilde{b}\tilde{c}}(\hat{\bar{\Gamma}})$  its torsion.

On a space-time  $E$  we also define the second affine connection  $\overline{W}^\alpha_\beta$  such that

$$\overline{W}^\alpha_\beta = \overline{\omega}^\alpha_\beta - \frac{2}{3} \delta^\alpha_\beta \overline{W}, \quad (2.75)$$

where

$$\overline{W} = \overline{W}_\gamma \overline{\theta}^\gamma = \frac{1}{2} (\overline{W}^\sigma_{\gamma\sigma} - \overline{W}^\sigma_{\gamma\sigma}).$$

We proceed a nonsymmetric metrization of a principal fiber bundle  $\underline{P}$  according to (2.51). Thus we define a right-invariant connection with respect to an action of the group  $H$  compatible with a tensor  $\kappa_{\tilde{A}\tilde{B}}$

$$\begin{aligned} D\kappa_{\tilde{A}\tilde{B}} &= \kappa_{\tilde{A}\tilde{D}} Q^{\tilde{D}}_{\tilde{B}\tilde{C}}(\Gamma) \theta^{\tilde{C}} \\ Q^{\tilde{D}}_{\tilde{B}\tilde{D}}(\Gamma) &= 0 \end{aligned} \quad (2.76)$$

where  $\omega^{\tilde{A}}_{\tilde{B}} = \Gamma^{\tilde{A}}_{\tilde{B}\tilde{C}} \tilde{\theta}^{\tilde{C}}$ .  $D$  is an exterior covariant derivative with respect to the connection  $\omega^{\tilde{A}}_{\tilde{B}}$  and  $Q^{\tilde{A}}_{\tilde{B}\tilde{C}}$  its torsion. After some calculations one finds

$$\omega^{\tilde{A}}_{\tilde{B}} = \left( \frac{\pi^*(\overline{\omega}^{\tilde{A}}_{\tilde{B}}) - \ell_{db} \gamma^{MA} L^d_{MB} \theta^b}{\ell_{bd} \gamma^{AB} (2H^d_{CB} - L^d_{CB}) \theta^C} \middle| \frac{L^a_{BC} \theta^C}{\tilde{\omega}^a_b} \right) \quad (2.77)$$

where

$$L^d_{MB} = -L^d_{BM} \quad (2.78)$$

$$\ell_{dc} \gamma_{MB} \gamma^{CM} L^d_{CA} + \ell_{cd} \gamma_{AM} \gamma^{MC} L^d_{BC} = 2 \ell_{cd} \gamma_{AM} \gamma^{MC} H^d_{BC}, \quad (2.79)$$

$L^d_{CA}$  is Ad-type tensor with respect to  $H$  (Ad-covariant on  $\underline{P}$ )

$$\tilde{\omega}^a_b = \tilde{\Gamma}^a_{bc} \theta^c \quad (2.80)$$

$$\ell_{db} \tilde{\Gamma}^d_{ac} + \ell_{ad} \tilde{\Gamma}^d_{cb} = -\ell_{db} C^d_{ac} \quad (2.81)$$

$$\tilde{\Gamma}^d_{ac} = -\tilde{\Gamma}^d_{ca}, \quad \tilde{\Gamma}^d_{ad} = 0. \quad (2.82)$$

We define on  $P$  a second connection

$$W^{\tilde{A}}_{\tilde{B}} = \omega^{\tilde{A}}_{\tilde{B}} - \frac{4}{3(m+2)} \delta^{\tilde{A}}_{\tilde{B}} \overline{W}. \quad (2.83)$$

Thus we have on  $P$  all  $(m+4)$ -dimensional analogues of geometrical quantities from NGT, i.e.

$$W^{\tilde{A}}_{\tilde{B}}, \quad \omega^{\tilde{A}}_{\tilde{B}} \quad \text{and} \quad \kappa_{\tilde{A}\tilde{B}}.$$

### 3 GSW (Glashow–Salam–Weinberg) model in the Nonsymmetric Kaluza–Klein Theory

Let  $\underline{P}$  be a principal fiber bundle

$$\underline{P} = (P, V, \pi, H, H) \quad (3.1)$$

over the base space  $V = E \times S^2$  (where  $E$  is a space-time,  $S^2$ —a two-dimensional sphere) with a projection  $\pi$ , a structural group  $H$ , a typical fiber  $H$  and a bundle manifold  $P$ . We suppose that  $H$  is semisimple. Let us define on  $P$  a connection  $\omega$  which has values in a Lie algebra of  $H$ ,  $\mathfrak{h}$ . Let us suppose that a group  $\text{SO}(3)$  is acting on  $S^2$  in a natural way. We suppose that  $\omega$  is invariant with respect to an action of the group  $\text{SO}(3)$  on  $V$  in such a way that this action is equivalent to  $\text{SO}(3)$  action on  $S^2$ . This is equivalent to the condition (2.56). If we take a section  $e : E \rightarrow P$  we get

$$e^* \omega = A^a_A \bar{\theta}^A X_a = A_A \bar{\theta}^A \quad (3.2)$$

where  $\bar{\theta}^A$  is a frame on  $V$  and  $X_a$  are generators of the Lie algebra  $\mathfrak{h}$ .

$$[X_a, X_b] = C^c_{ab} X_c. \quad (3.3)$$

We define a curvature of the connection  $\omega$

$$\Omega = d\omega + \frac{1}{2}[\omega, \omega]. \quad (3.4)$$

Taking a section  $e$

$$e^* \Omega = \frac{1}{2} F^a_{AB} \bar{\theta}^A \wedge \bar{\theta}^B X_a = \frac{1}{2} F_{AB} \bar{\theta}^A \wedge \bar{\theta}^B \quad (3.5)$$

$$F^a_{AB} = \partial_A A^a_B - \partial_B A^a_A - C^a_{cb} A^c_A A^c_B. \quad (3.6)$$

Let us consider a local coordinate systems on  $V$ . One has  $x^A = (x^\mu, \psi, \varphi)$  where  $x^\mu$  are coordinate system on  $E$ ,  $\bar{\theta}^\mu = dx^\mu$ , and  $\psi$  and  $\varphi$  are polar and azimuthal angles on  $S^2$ ,  $\bar{\theta}^5 = d\psi$ ,  $\bar{\theta}^6 = d\varphi$ . We have  $A, B, C = 1, 2, \dots, 6$ ,  $\mu = 1, 2, 3, 4$ . Let us introduce vector fields on  $V$  corresponding to the infinitesimal action of  $\text{SO}(3)$  on  $V$  (see Ref. [40]). These vector fields are called  $\delta_m = (\delta_m^A)$ ,  $\bar{m} = 1, 2, 3$ ,  $A = 1, 2, \dots, 6$ . Moreover, they are acting only on the last two dimensions ( $A, B = 5, 6$ ,  $\bar{a}, \bar{b} = 5, 6$ ). We get:

$$\begin{aligned} \delta_m^\mu &= 0 & \text{and} \\ \delta_1^\psi &= \cos \varphi, & \delta_1^\varphi &= -\cot \psi \sin \varphi, \\ \delta_2^\psi &= -\sin \varphi, & \delta_2^\varphi &= -\cot \psi \cos \varphi, \\ \delta_3^\psi &= 0, & \delta_3^\varphi &= 1. \end{aligned} \quad (3.7)$$

They satisfy commutation relation of the Lie algebra  $A_1$  of a group  $\text{SO}(3)$ ,

$$\delta_{\bar{m}}^A \partial_A \delta_{\bar{n}}^B - \delta_{\bar{n}}^A \partial_A \delta_{\bar{m}}^B = \varepsilon_{\bar{m}\bar{n}\bar{p}} \delta_{\bar{p}}^B. \quad (3.8)$$

The gauge field  $A_A$  is spherically symmetric (invariant with respect to an action of a group  $\text{SO}(3)$ ) iff for some  $V_{\bar{m}}$ —a field on  $V$  with values in the Lie algebra  $\mathfrak{h}$ —

$$\partial_B \delta_{\bar{m}}^A A_A + \delta_{\bar{m}}^A \partial_A A_B = \partial_B V_{\bar{m}} - [A_B, V_{\bar{m}}]. \quad (3.9)$$

It means that

$$\mathcal{L}_{\delta_{\bar{m}}} A_A = \partial_B V_{\bar{m}} - [A_A, V_{\bar{m}}], \quad (3.10)$$

a Lie derivative of  $A_A$  with respect to  $\delta_{\bar{m}}$  results in a gauge transformation (see also Eq. (2.56)).



Eq. (3.10) is satisfied if

$$V_1 = \Phi_3 \frac{\sin \varphi}{\sin \psi}, \quad V_2 = \Phi_3 \frac{\cos \varphi}{\sin \psi}, \quad V_3 = 0 \quad (3.11)$$

and

$$A_\mu = A_\mu(x), \quad A_\psi = -\Phi_1(x) = A_5 = \Phi_5, \quad A_\varphi = \Phi_2(x) \sin \psi - \Phi_3 \cos \psi = A_6 = \Phi_6 \quad (3.12)$$

with the following constraints

$$\begin{aligned} [\Phi_3, \Phi_1] &= -\Phi_2, \\ [\Phi_3, \Phi_2] &= \Phi_1, \\ [\Phi_3, A_\mu] &= 0. \end{aligned} \quad (3.13)$$

$A_\mu, \Phi_1, \Phi_2$  are fields on  $E$  with values in the Lie algebra of  $H(\mathfrak{h})$ ,  $\Phi_3$  is a constant element of Cartan subalgebra of  $\mathfrak{h}$ . Let us introduce some additional elements according to the Nonsymmetric Hermitian Kaluza–Klein Theory. According to Section 2 we have on  $E$  a nonsymmetric Hermitian tensor  $g_{\mu\nu}$ , connections  $\overline{\omega}^\alpha_\beta$  and  $\overline{W}^\alpha_\beta$ . On  $S^2$  we have a nonsymmetric metric tensor

$$\gamma_{\tilde{a}\tilde{b}} = r^2 g_{\tilde{a}\tilde{b}} = r^2 (h^0_{\tilde{a}\tilde{b}} + \zeta k^0_{\tilde{a}\tilde{b}}) \quad (3.14)$$

where  $r$  is the radius of a sphere  $S^2$  and  $\zeta$  is considered to be pure imaginary,

$$h^0_{\tilde{a}\tilde{b}} = \left( \begin{array}{c|c} -1 & 0 \\ \hline 0 & -\sin^2 \psi \end{array} \right) \quad (3.15)$$

$$k^0_{\tilde{a}\tilde{b}} = \left( \begin{array}{c|c} 0 & \sin \psi \\ \hline -\sin \psi & 0 \end{array} \right) \quad (3.16)$$

and a connection compatible with this nonsymmetric metric

$$g_{\tilde{a}\tilde{b}} = \begin{array}{c} 5 \qquad 6 \\ \left( \begin{array}{c|c} -1 & \zeta \sin \psi \\ \hline -\zeta \sin \psi & -\sin^2 \psi \end{array} \right) \begin{array}{c} 5 \\ 6 \end{array} \end{array} \quad (3.17)$$

$$\tilde{g} = \det(g_{\tilde{a}\tilde{b}}) = \sin^2 \psi (1 + \zeta^2) \quad (3.18)$$

$$g^{\tilde{a}\tilde{b}} = \frac{1}{\sin^2 \psi (1 + \zeta^2)} \begin{array}{c} 5 \qquad 6 \\ \left( \begin{array}{c|c} -\sin^2 \psi & -\zeta \sin \psi \\ \hline \zeta \sin \psi & -1 \end{array} \right) \begin{array}{c} 5 \\ 6 \end{array} \end{array}, \quad (3.19)$$

$\tilde{a}, \tilde{b} = 5, 6$ . In this way we have to do with Kählerian structure on  $S^2$  (Riemannian, symplectic and complex which are compatible). This seems to be very interesting in further research connecting unification of all fundamental interactions. On  $H$  we define a nonsymmetric metric

$$\ell_{ab} = h_{ab} + \xi k_{ab} \quad (3.20)$$

where  $k_{ab}$  is a right-invariant skew-symmetric 2-form on  $H$ .

One can rewrite the constraints (3.13) in the form

$$\begin{aligned} [\Phi_3, \Phi] &= i\Phi \\ [\Phi_3, \tilde{\Phi}] &= -i\tilde{\Phi} \\ [\Phi_3, A_\mu] &= 0 \end{aligned} \quad (3.21)$$

where  $\Phi = \Phi_1 + i\Phi_2$ ,  $\tilde{\Phi} = \Phi_1 - i\Phi_2$  (see Ref. [40]).

In this way our 6-dimensional gauge field (a connection on a fiber bundle) has been reduced to a 4-dimensional gauge one (a connection on a fiber bundle over a space-time  $E$ ) and a collection of scalar fields defined on  $E$  satisfying some constraints. According to our approach there is defined on  $S^2$  a nonsymmetric connection compatible with a nonsymmetric tensor  $g_{\tilde{a}\tilde{b}}$ ,  $\tilde{a}, \tilde{b} = 5, 6$ ,

$$\begin{aligned}\hat{D}g_{\tilde{a}\tilde{b}} &= g_{\tilde{a}\tilde{d}}Q_{\tilde{b}\tilde{c}}^{\tilde{d}}(\hat{\Gamma})\bar{\theta}^{\tilde{c}} \\ Q_{\tilde{b}\tilde{d}}^{\tilde{d}}(\tilde{\Gamma}) &= 0\end{aligned}\tag{3.22}$$

where  $\hat{D}$  is an exterior covariant derivative with respect to a connection  $\hat{\omega}_{\tilde{b}}^{\tilde{a}} = \hat{\Gamma}_{\tilde{b}\tilde{c}}^{\tilde{a}}\bar{\theta}^{\tilde{c}}$  and  $Q_{\tilde{b}\tilde{c}}^{\tilde{d}}(\hat{\Gamma})$  its torsion.

Let us metrize a bundle  $P$  in a nonsymmetric way. On  $V$  we have nonsymmetric tensor (see Ref. [1])

$$\gamma_{AB} = \left( \begin{array}{c|c} g_{\mu\nu} & 0 \\ \hline 0 & r^2 g_{\tilde{a}\tilde{b}} \end{array} \right)\tag{3.23}$$

and a nonsymmetric connection  $\varpi^A_B = \Gamma^A_{BC}\theta^C$  compatible with this tensor

$$\begin{aligned}\overline{D}\gamma_{AB} &= \gamma_{AD}Q^D_{BC}(\overline{\Gamma})\theta^C \\ Q^D_{BD}(\overline{\Gamma}) &= 0.\end{aligned}\tag{3.24}$$

The form of this connection is as follows

$$\varpi^A_B = \left( \begin{array}{c|c} \overline{\omega}^\alpha_\beta & 0 \\ \hline 0 & \hat{\omega}^{\tilde{a}}_{\tilde{b}} \end{array} \right)\tag{3.25}$$

where  $\overline{D}$  is an exterior covariant derivative with respect to  $\overline{\omega}^A_B$  and  $Q^D_{BC}(\overline{\Gamma})$  its torsion.

Afterwards we define on  $P$  a nonsymmetric tensor

$$\kappa_{\tilde{A}\tilde{B}}\theta^{\tilde{A}} \otimes \theta^{\tilde{B}} = \pi^*(\gamma_{AB}\bar{\theta}^A \otimes \theta^B) + \ell_{ab}\theta^a \otimes \theta^b\tag{3.26}$$

where

$$\theta^{\tilde{A}} = (\pi^*(\bar{\theta}^A), \lambda\omega^a),\tag{3.27}$$

$\omega = \omega^0 X_a$  is a connection defined on  $P$  ( $\tilde{A}, \tilde{B}, \tilde{C} = 1, 2, \dots, n+6$ ).

We define on  $P$  two connections  $\omega^A_B$  and  $W^A_B$  such that  $\omega^A_B$  is compatible with a nonsymmetric tensor  $\kappa_{\tilde{A}\tilde{B}}$ ,

$$\begin{aligned}D\kappa_{\tilde{A}\tilde{B}} &= \kappa_{\tilde{A}\tilde{D}}Q^{\tilde{D}}_{\tilde{B}\tilde{C}}(\Gamma)\theta^{\tilde{C}} \\ Q^{\tilde{D}}_{\tilde{B}\tilde{D}}(\Gamma) &= 0,\end{aligned}\tag{3.28}$$

where  $D$  is an exterior covariant derivative with respect to a connection  $\omega^{\tilde{A}}_{\tilde{B}}$  and  $Q^{\tilde{D}}_{\tilde{B}\tilde{C}}(\Gamma)$  its torsion.

The second connection

$$W^{\tilde{A}}_{\tilde{B}} = \omega^{\tilde{A}}_{\tilde{B}} - \frac{4}{3(n+4)}\delta^{\tilde{A}}_{\tilde{B}}\overline{W} \quad (n = \dim H).\tag{3.29}$$

In this way we have all quantities known from Section 2. We calculate a scalar of curvature (Moffat–Ricci) for a connection  $W^{\tilde{A}}_{\tilde{B}}$  and afterwards an action

$$\begin{aligned} S &= -\frac{1}{V_1 V_2} \int_U \sqrt{-g} d^4 x \int_H \sqrt{|\ell|} d^n x \int_{S^2} \sqrt{|\tilde{g}|} d\Omega R(W) \\ &= -\frac{1}{r^2 V_1 V_2} \int_U \sqrt{-g} d^4 x \int_{S^2} \sqrt{|\tilde{g}|} d\Omega \left( \bar{R}(\bar{W}) \right. \\ &\quad \left. + \frac{8\pi G_N}{c^4} \left( \mathcal{L}_{\text{YM}} + \frac{1}{4\pi r^2} \mathcal{L}_{\text{kin}}(\nabla\Phi) - \frac{1}{8\pi r^2} V(\Phi) - \frac{1}{2\pi r^2} \mathcal{L}_{\text{int}}(\Phi, \tilde{A}) \right) + \lambda_c \right) \end{aligned} \quad (3.30)$$

where  $V_1 = \int_U \sqrt{|\ell|} d^n x$ ,  $V_2 = \int_{S^2} \sqrt{|\tilde{g}|} d\Omega$ ,  $U \subset E$ ,

$$\lambda_c = \left( \frac{\alpha_s^2}{\ell_{\text{pl}}^2} \tilde{R}(\tilde{\Gamma}) + \frac{1}{r^2} \tilde{P} \right) \quad (3.31)$$

where  $\tilde{R}(\tilde{\Gamma})$  is a Moffat–Ricci curvature scalar on a group  $H$  (see Section 3 for details).

$$\tilde{P} = \frac{1}{V_2} \int_{S^2} \sqrt{|\tilde{g}|} d\Omega \hat{R}(\hat{\Gamma}) \quad (3.32)$$

where  $\hat{R}(\hat{\Gamma})$  is a Moffat–Ricci curvature scalar on  $S^2$  for a connection  $\hat{\omega}_{\tilde{b}}^{\tilde{a}}$ .

$$\mathcal{L}_{\text{YM}} = -\frac{1}{8\pi} \ell_{ij} (\tilde{H}^{(i} \tilde{H}^{j)} - \tilde{L}^{i\mu\nu} \tilde{H}^j_{\mu\nu}) \quad (3.33)$$

where

$$\ell_{ij} g_{\mu\beta} g^{\gamma\mu} \tilde{L}^i_{\gamma\alpha} + \ell_{ji} g_{\alpha\mu} g^{\mu\gamma} \tilde{L}^i_{\beta\gamma} = 2\ell_{ji} g_{\alpha\mu} g^{\mu\gamma} \tilde{H}^i_{\beta\gamma} \quad (3.34)$$

One gets from (2.78)

$$L^b_{\tilde{a}\tilde{b}} = h^{bc} \ell_{cd} H^d_{\tilde{b}\tilde{a}}, \quad (3.35)$$

$$\begin{aligned} V(\Phi) &= -\frac{1}{V_2} \int \sqrt{|\tilde{g}|} d\Omega (2h_{cd} (H^c_{\tilde{a}\tilde{b}} g^{\tilde{a}\tilde{b}}) (H^d_{\tilde{c}\tilde{d}} g^{\tilde{c}\tilde{d}}) - \ell_{cd} g^{\tilde{a}\tilde{m}} g^{\tilde{b}\tilde{n}} L^c_{\tilde{a}\tilde{b}} H^d_{\tilde{m}\tilde{n}}) \\ &= \frac{1}{V_2} \frac{2\pi^2}{\sqrt{1+\zeta^2}} \kappa((\varepsilon_{\tilde{r}\tilde{s}\tilde{t}} \Phi_{\tilde{t}} + [\Phi_{\tilde{r}}, \Phi_{\tilde{s}}]), (\varepsilon_{\tilde{r}\tilde{s}\tilde{t}} \Phi_{\tilde{t}} + [\Phi_{\tilde{r}}, \Phi_{\tilde{s}}])) \end{aligned} \quad (3.36)$$

$$\kappa_{de} = (1 - 2\zeta^2) h_{de} + \xi^2 k^c_d k_{ce} \quad (3.37)$$

where

$$k^c_d = h^{cf} k_{fd} \quad (3.38)$$

$$V_2 = \int_{S^2} \sqrt{|\tilde{g}|} d\Omega = 4\pi \sqrt{1+\zeta^2}, \quad (3.39)$$

$\tilde{r}, \tilde{s}, \tilde{t} = 1, 2, 3$ ,  $\varepsilon_{\tilde{r}\tilde{s}\tilde{t}}$  is a usual antisymmetric symbol  $\varepsilon_{123} = 1$ .

We get also from (2.78)

$$\ell_{dc} g_{\mu\beta} g^{\gamma\mu} L^d_{\gamma\tilde{a}} + \ell_{cd} L^d_{\beta\tilde{a}} = 2\ell_{cd} F^c_{\beta\tilde{a}}. \quad (3.40)$$

Using the equation (see Ref. [5])

$$\begin{aligned}
L^n_{\omega\tilde{m}} = & \nabla_\omega^{\text{gauge}} \Phi_{\tilde{m}}^n + \xi k_d^n \nabla_\omega^{\text{gauge}} \Phi_{\tilde{m}}^d - (\zeta \nabla_\omega^{\text{gauge}} \Phi_{\tilde{a}}^n h^{0\tilde{a}\tilde{d}} k_{0\tilde{d}\tilde{m}} + \tilde{g}^{(\alpha\mu)} \nabla_\alpha^{\text{gauge}} \Phi_{\tilde{m}}^n g_{[\mu\omega]}) \\
& - 2\xi \zeta k_d^n \nabla_\omega^{\text{gauge}} \Phi_{\tilde{a}}^d \tilde{g}^{(\delta\alpha)} g_{[\alpha\omega]} h^{0\tilde{d}\tilde{a}} k_{\tilde{a}\tilde{m}}^0 + \xi k_d^n (\zeta^2 h^{\tilde{d}\tilde{a}} \nabla_\omega^{\text{gauge}} \Phi_{\tilde{a}}^d k_{\tilde{d}\tilde{b}}^0 k_{\tilde{m}\tilde{c}}^0 h^{0\tilde{c}\tilde{b}} \\
& + \nabla_\beta^{\text{gauge}} \Phi_{\tilde{m}}^d \tilde{g}^{(\delta\beta)} g_{[\delta\alpha]} g_{[\omega\mu]} \tilde{g}^{(\alpha\mu)}) - \xi^2 k^{nb} k_{bd} (\zeta \nabla_\omega^{\text{gauge}} \Phi_{\tilde{a}}^d h^{0\tilde{a}\tilde{b}} k_{\tilde{m}\tilde{b}}^0 + \tilde{g}^{(\alpha\beta)} \nabla_a^{\text{gauge}} \Phi_{\tilde{m}}^d g_{[\omega\beta]})
\end{aligned}$$

where

$$k^{nb} = h^{na} h^{bp} k_{ap},$$

one gets

$$\begin{aligned}
L^n_{\omega\tilde{m}} = & \nabla_\omega^{\text{gauge}} \Phi_{\tilde{m}}^n + \xi k_d^n \nabla_\omega^{\text{gauge}} \Phi_{\tilde{m}}^d - \tilde{g}^{(\alpha\mu)} \nabla_\alpha^{\text{gauge}} \Phi_{\tilde{m}}^n g_{[\mu\omega]} \\
& + \xi k_d^n \nabla_\beta^{\text{gauge}} \Phi_{\tilde{m}}^d \tilde{g}^{(\delta\beta)} g_{[\delta\alpha]} g_{[\omega\mu]} \tilde{g}^{(\alpha\mu)} - \xi^2 k^{nb} k_{bd} \tilde{g}^{(\alpha\beta)} \nabla^{\text{gauge}} \Phi_{\tilde{m}}^d g_{[\omega\beta]}. \quad (3.41)
\end{aligned}$$

Moreover, now we have to do with Minkowski space  $g_{\mu\nu} = \eta_{\mu\nu}$  and

$$L^n_{\omega\tilde{m}} = H^n_{\omega\tilde{m}} + \xi k_d^n H^d_{\omega\tilde{m}}. \quad (3.42)$$

We remember that  $\tilde{m} = 5, 6$  or  $\varphi, \psi$  and that

$$H^n_{\mu\tilde{m}} = \nabla_\mu^{\text{gauge}} \Phi_{\tilde{m}}^n. \quad (3.43)$$

We have

$$\mathcal{L}_{\text{kin}}(H^n_{\mu\tilde{m}}) = \frac{1}{V_2} \int \sqrt{|\tilde{g}|} d\Omega (\ell_{ab} \eta^{\beta\mu} L^a_{\beta\tilde{b}} H^b_{\mu\tilde{a}} \tilde{g}^{\tilde{b}\tilde{a}}). \quad (3.44)$$

Finally we get

$$\mathcal{L}_{\text{kin}}(\nabla_\mu \Phi_{\tilde{m}}) = \frac{2\pi^2}{V_2} \frac{\eta^{\mu\nu}}{\sqrt{1+\zeta^2}} \bar{\kappa} (\nabla_\mu^{\text{gauge}} \Phi_{\tilde{m}}, \nabla_\nu^{\text{gauge}} \Phi_{\tilde{m}}) \quad (3.45)$$

$$\bar{\kappa}_{ad} = (h_{ad} + \xi^2 k_{ab} k^b_d) \quad (3.46)$$

where

$$\nabla_\mu^{\text{gauge}} \Phi_{\tilde{m}} = \partial_\mu \Phi_{\tilde{m}}^a - [A_\mu, \Phi_{\tilde{m}}]. \quad (3.47)$$

Now we follow Ref. [40] and suppose  $\text{rank } H = 2$  and afterwards  $H = G2$ . In this way our lagrangian can go to the GSW model where  $\text{SU}(2) \times \text{U}(1)$  is a little group of  $\Phi_3$  (see Appendix B). We get also a Higgs' field complex doublet and spontaneous symmetry breaking and mass generation for intermediate bosons. For simplicity we take  $\xi = 0$  and also we do not consider an influence of the nonsymmetric gravity on a Higgs' field. We get also a mixing angle  $\theta_W$  (Weinberg angle). If we choose  $H = G2$  we get  $\theta_W = 30^\circ$ . We get also some predictions of masses

$$\frac{M_H}{M_W} = \frac{1}{\cos \theta_W} \cdot \sqrt{1 - 2\zeta^2} \quad (3.48)$$

where  $\zeta$  is an arbitrary constant

$$\frac{M_H}{M_W} = \frac{2\sqrt{1 - 2\zeta^2}}{\sqrt{3}}. \quad (3.49)$$

We take  $M_H \simeq 125 \text{ GeV}$  and  $M_W \simeq 80 \text{ GeV}$  (see Refs [48, 49, 50, 51, 52]).

One gets

$$\zeta = \pm 0.911622i. \quad (3.50)$$

Thus  $\zeta$  is pure imaginary. This means we can explain mass pattern in GSW model.  $r$  gives us a scale of mass and is an arbitrary parameter.

Moreover, a scale of energy is equal to  $M = \frac{hc}{r\sqrt{2\pi}\sqrt{1+\zeta^2}}$  which we equal to MEW (electroweak) energy scale, i.e. to  $M_W$ . One gets  $r \simeq 2.39 \times 10^{-18} \text{ m}$ . In the original Manton model Higgs' boson is too light. We predict here masses for  $W, Z^0$  and Higgs bosons in the theory taking two parameters,  $\zeta$  (Eq. (3.50)) and  $r \simeq 2.39 \times 10^{-18} \text{ m}$  in order to get desired pattern of masses. The value of the Weinberg angle derived here for  $H = G2$  has nothing to do with "GUT driven" value  $\frac{1}{4}$  for  $\frac{1}{4}$  is a value of our  $\sin^2 \theta_W$ , not  $\sin \theta_W$ . According to Ref. [40] a Lie group  $H$  should have a Lie algebra  $\mathfrak{h}$  with rank 2. We have only three possibilities:  $G2$ ,  $SU(3)$  and  $SO(5)$ . The angle between two roots plays a role of a Weinberg angle. For  $SO(5)$   $\theta = 45^\circ$  and for  $SU(3)$   $\theta = 60^\circ$ . Only for  $G2$ ,  $\theta = \theta_W = 30^\circ$ , which is close to the experimental value. In this way a unification chooses  $H = G2$ .

Let us notice that  $\dim G2 = 14$  and for this  $\dim P = 20$ .

Moreover, we have

$$M_Z = \frac{M_W}{\cos \theta} = \frac{M_W}{\cos \theta_W} = \frac{2}{\sqrt{3}} M_W \simeq 92.4 \quad (3.51)$$

and we get from the theory

$$\sin^2 \theta_W = 0.25 \quad (\theta_W = 30^\circ). \quad (3.52)$$

However from the experiment we get

$$\sin^2 \theta_W = 0.2397 \pm 0.0013 \quad (3.53)$$

which is not 0.25.

Moreover, from theoretical point of view the value 0.25 is a value without radiation corrections and it is possible to tune it at  $Q = 91.2 \text{ GeV}/c$  in the  $\overline{\text{MS}}$  scheme to get the desired value.

Let us notice the following fact. In the electroweak theory we have a Lagrangian for neutral current interaction

$$\mathcal{L}_N = qJ_\mu^{\text{em}} A^\mu + \frac{g}{\cos \theta_W} (J_\mu^3 - \sin^2 \theta_W J_\mu^{\text{em}}) Z^{0\mu} = qJ_\mu^{\text{em}} A^\mu + \sum_f \bar{\psi}_f \gamma_\mu (g_V^f - g_A^f \gamma^5) \psi_f Z^{0\mu} \quad (3.54)$$

where  $g_V^f$  and  $g_A^f$  are coupling constants for vector and axial interactions for a fermion  $f$ . One gets

$$\begin{aligned} g_V^f &= \frac{2q}{\sin 2\theta_W} (T_f^3 - 2q_f \sin^2 \theta_W) \\ g_A^f &= \frac{2q}{\sin 2\theta_W} \end{aligned} \quad (3.55)$$

where  $T_f^3$  is the third component of a weak isospin of a fermion  $f$  and  $q_f$  is its electric charge measured in elementary charge  $q$ ,

$$q_f = T_f^3 + \frac{Y_f}{2} \quad (3.56)$$

where  $Y_f$  is a weak hypercharge for  $f$ . It is easy to see that for an electron we get  $g_V^f = 0$  if  $\theta_W = 30^\circ$ .

Moreover, we know from the experiment that

$$g_V^f \neq 0 \quad (3.57)$$

(see Ref. [48]).

Following Ref. [40] we use the following formulae

$$\Phi_5 = \frac{1}{2}(\varphi_1^* x_{-\alpha} + \varphi_2^* x_{-\beta} - \varphi_1 x_\alpha - \varphi_2 x_\beta) \quad (3.58)$$

$$\Phi_6 = \frac{\sin \psi}{2i}(\varphi_1 x_\alpha + \varphi_2 x_\beta + \varphi_1^* x_{-\alpha} + \varphi_2^* x_{-\beta}) - \Phi_3 \cos \psi. \quad (3.59)$$

$\Phi_3$  is constant and commutes with a reduced connection.  $SU(2) \times U(1)$  is a little group of  $\Phi_3$ ,

$$\Phi_3 = \frac{1}{2}i(2 - \langle \gamma, \alpha \rangle)^{-1}(h_\alpha + h_\beta), \quad (3.60)$$

$x_\alpha, x_{-\alpha}, x_\beta, x_{-\beta}$  are elements of a Lie algebra  $\mathfrak{h}$  of  $H$  (see Ref. [53]) corresponding to roots  $\alpha, -\alpha, \beta, -\beta$ ,  $h_\alpha$  and  $h_\beta$  are elements of Cartan subalgebra of  $\mathfrak{h}$  such that

$$h_\alpha = \frac{2\alpha_i}{\alpha \cdot \alpha} H_i = [x_\alpha, x_{-\alpha}], \quad (3.61)$$

where  $\alpha = (\alpha_1, \dots, \alpha_k)$ ,  $k = \text{rank}(\mathfrak{h})$ ,  $\gamma = \alpha - \beta$ ,  $[H_i, x_\omega] = \omega_i x_\omega$ ,  $H_i$  form Cartan subalgebra of  $\mathfrak{h}$ ,  $[x_\omega, x_\tau] = C_{\omega, \tau} x_{\omega+\tau}$  if  $\omega + \tau$  is a root, if  $\omega + \tau$  is not a root  $x_\omega$  and  $x_\tau$  commute. We take  $k = 2$ .

$$\langle \gamma, \alpha \rangle = \frac{2\gamma \cdot \alpha}{\alpha \cdot \alpha} = 2 \frac{|\gamma|}{|\alpha|} \cos \theta. \quad (3.62)$$

In this way we get a Higgs' doublet  $(\varphi_1^1)_{\varphi_2} = \tilde{\varphi}$ .

The  $SU(2) \times U(1)$  generators are given by

$$\begin{aligned} t_1 &= \frac{1}{2}i(x_\gamma + x_{-\gamma}) \\ t_2 &= \frac{1}{2}(x_\gamma - x_{-\gamma}) \\ t_3 &= \frac{1}{2}ih_\gamma \\ y &= \frac{1}{2}ih. \end{aligned} \quad (3.63)$$

$h$  is an element of Cartan subalgebra orthogonal to  $h_\gamma$  with the same norm. Now everything is exactly the same as in Ref. [40] except the fact that

$$\bar{k}_{ad} = h_{ad} - \xi^2 k_{ab} k_d^b \quad (3.64)$$

$$k_{ad} = (1 - 2\xi^2)h_{ad} - \xi^2 k_{ab} k_d^b. \quad (3.65)$$

In Ref. [40]

$$\bar{k}_{ad} = k_{ad} = h_{ad}. \quad (3.66)$$

A four-potential of Yang–Mills' field (a connection  $\omega_E$ ) can be written as

$$A_\mu = \sum_{i=1}^3 A_\mu t_i + B_\mu y \quad (3.67)$$

$$\text{or } A_\mu = \frac{1}{2} i(A_\mu^- x_\gamma + A_\mu^+ x_{-\gamma} + A_\mu^3 h_\gamma + B_\mu h) \quad (3.68)$$

$$A_\mu^\pm = A_\mu^1 \pm i A_\mu^2. \quad (3.69)$$

We have (see Ref. [40])

$$\begin{aligned} h(t_i, t_j) &= -\frac{1}{\gamma \cdot \gamma} \delta_{ij} \\ h(y, y) &= -\frac{1}{\gamma \cdot \gamma} \\ h(t_i, y) &= 0 \\ F_{\mu\nu} &= (\partial_\mu A_\nu^a - \partial_\nu A_\mu^a + \varepsilon_{bc}^a A_\mu^b A_\nu^c) t_a + (\partial_\mu B_\nu - \partial_\nu B_\mu) y = F_{\mu\nu}^a t_a + B_{\mu\nu} y \end{aligned} \quad (3.70)$$

$$h(F_{\mu\nu}, F_{\mu\nu}) = -\frac{\delta_{ab}}{\gamma \cdot \gamma} F_{\mu\nu}^a F^{b\mu\nu} - \frac{1}{\gamma \cdot \gamma} B_{\mu\nu} B^{\mu\nu} \quad (3.71)$$

$$\begin{aligned} \nabla_\mu^{\text{gauge}} \Phi &= \left( \partial_\mu \varphi_1 - \frac{1}{2} i A_\mu^- \varphi_2 - \frac{1}{2} i A_\mu^3 \varphi_1 - \frac{1}{2} i \tan \theta B_\mu \varphi_1 \right) x_\alpha \\ &\quad + \left( \partial_\mu \varphi_2 - \frac{1}{2} i A_\mu^+ \varphi_1 + \frac{1}{2} i A_\mu^3 \varphi_2 - \frac{1}{2} i \tan \theta B_\mu \varphi_2 \right) x_\beta \end{aligned} \quad (3.72)$$

$$\begin{aligned} \nabla_\mu^{\text{gauge}} \tilde{\Phi} &= -\left( \partial_\mu \varphi_1^* + \frac{1}{2} i A_\mu^+ \varphi_2^* + \frac{1}{2} i A_\mu^3 \varphi_1^* + \frac{1}{2} i \tan \theta B_\mu \varphi_1^* \right) x_{-\alpha} \\ &\quad - \left( \partial_\mu \varphi_2^* + \frac{1}{2} i A_\mu^- \varphi_1^* - \frac{1}{2} i A_\mu^3 \varphi_2^* + \frac{1}{2} i \tan \theta B_\mu \varphi_2^* \right) x_{-\beta} \end{aligned} \quad (3.73)$$

We redefine the fields  $A_\mu^a$ ,  $B_\mu$  and  $\tilde{\varphi}$  with some rescaling ( $g$  is a coupling constant)

$$A_\mu'^a = L_1 A_\mu^a, \quad B_\mu' = L_1 B_\mu, \quad \tilde{\varphi}' = L_2 \tilde{\varphi} \quad (3.74)$$

where

$$L_1 = \frac{1}{g} \frac{1}{(\gamma \cdot \gamma)^{1/2}} \quad (3.75)$$

$$L_2 = \frac{1}{g} \left( \frac{\gamma \cdot \gamma}{\alpha \cdot \alpha} \right)^{1/2} \quad (3.76)$$

We proceed the following transformation

$$\begin{pmatrix} Z_\mu^0 \\ A_\mu \end{pmatrix} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} A_\mu^3 \\ B_\mu \end{pmatrix}. \quad (3.77)$$

According to the classical results we also have  $\frac{g'}{g} = \tan \theta$ , assuming  $q = g \sin \theta$ , where  $q$  is an elementary charge and  $g$  and  $g'$  are coupling constants of  $A_\mu^a$  and  $B_\mu$  fields. The spontaneous symmetry breaking and Higgs' mechanism in the Manton model works classical if we take for minimum of the potential

$$\tilde{\varphi}_0 = \begin{pmatrix} 0 \\ \frac{v}{\sqrt{2}} \end{pmatrix} e^{i\alpha}, \quad \alpha \text{ arbitrary phase}, \quad (3.78)$$

and we parametrize  $\tilde{\varphi} = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}$  in the following way

$$\tilde{\varphi}(x) = \exp\left(i \frac{1}{2v} \sigma^a t^a(x)\right) \begin{pmatrix} 0 \\ \frac{v+H(x)}{\sqrt{2}} \end{pmatrix}. \quad (3.79)$$

For a vacuum state we take

$$\tilde{\varphi}_0 = \begin{pmatrix} 0 \\ \frac{v}{\sqrt{2}} \end{pmatrix}, \quad (3.80)$$

$t^a(x)$  and  $H(x)$  are real fields on  $E$ .  $t^a(x)$  has been “eaten” by  $A_\mu^a$ ,  $a = 1, 2$ , and  $Z_\mu^0$  fields making them massive.  $H(x)$  is our Higgs' field.  $\sigma^a$  are Pauli matrices.

In the formulae (3.64)–(3.65) we take  $\xi = 0$ . One gets in the Lagrangian mass terms:

$$M_W^2 W_\mu^+ W^{-\mu} + \frac{1}{2} M_Z^2 Z_\mu^0 Z^{0\mu} - \frac{1}{2} M_H^2 H^2,$$

where  $W_\mu^+ = A_\mu^+$ ,  $W_\mu^- = A_\mu^-$ , getting masses for  $W^\pm$ ,  $Z^0$  bosons and a Higgs boson (see Eqs (3.48)–(3.52)). For G2  $\langle \gamma, \alpha \rangle = 3$  and  $\theta = 30^\circ$ ,  $\theta$  is identified with the Weinberg angle  $\theta_W$ .

In order to proceed a Higgs' mechanism and spontaneous symmetry breaking in this model we use the following gauge transformation

$$\tilde{\varphi}(x) \mapsto U(x) \tilde{\varphi}(x) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v + H(x) \end{pmatrix}, \quad (3.81)$$

where

$$v = \frac{2\sqrt{2}}{rg} \cos \theta \quad (3.82)$$

a vacuum value of a Higgs field

$$U(x) = \exp\left(-\frac{1}{2v} t^a(x) \sigma^a\right). \quad (3.83)$$

$H(x)$  is the remaining scalar field after a symmetry breaking and a Higgs' mechanism. One gets

$$A_\mu \mapsto A_\mu^u = \text{ad}'_{U^{-1}(x)} A_\mu + U^{-1}(x) \partial_\mu U(x) \quad (3.84)$$

$$F_{\mu\nu} \mapsto F_{\mu\nu}^u = \text{ad}'_{U^{-1}(x)} F_{\mu\nu}. \quad (3.85)$$

Using some additional fields  $\Phi_1, \Phi_2, \Phi_3$  and also  $\Phi$  and  $\tilde{\Phi}$ , we can write  $\overset{\text{gauge}}{\nabla}_\mu \Phi_5$  and  $\overset{\text{gauge}}{\nabla}_\mu \Phi_6$  in terms of Higgs' fields  $\varphi_1$  and  $\varphi_2$ ,

$$\overset{\text{gauge}}{\nabla}_\mu \Phi_5 = \frac{1}{2} \overset{\text{gauge}}{\nabla}_\mu (\Phi + \tilde{\Phi}) = \frac{1}{2} \left[ \left( \partial_\mu \varphi_1 - \frac{1}{2} i A_\mu^- \varphi_2 - \frac{1}{2} i A_\mu^3 \varphi_1 - \frac{1}{2} i \tan \theta B_\mu \varphi_1 \right) x_\alpha \right]$$



$$\begin{aligned}
& + \left( \partial_\mu \varphi_2 - \frac{1}{2} i A_\mu^+ \varphi_1 + \frac{1}{2} i A_\mu^+ \varphi_2 - \frac{1}{2} i B_\mu \varphi_2 \tan \theta \right) x_\beta \\
& - \left( \partial_\mu \varphi_1^* + \frac{1}{2} i A_\mu^+ \varphi_2^* + \frac{1}{2} i A_\mu^3 \varphi_1^* + \frac{1}{2} i B_\mu \varphi_1^* \tan \theta \right) x_{-\alpha} \\
& - \left( \partial_\mu \varphi_2^* + \frac{1}{2} i A_\mu^- \varphi_1^* - \frac{1}{2} i A_\mu^3 \varphi_2^* + \frac{1}{2} i \tan \theta B_\mu \varphi_2^* \right) x_{-\beta} \Big]
\end{aligned} \tag{3.86}$$

$$\begin{aligned}
\nabla_\mu^{\text{gauge}} \Phi_6 &= \frac{\sin \psi}{2i} \nabla_\mu^{\text{gauge}} (\Phi - \tilde{\Phi}) = \frac{\sin \psi}{2i} \left[ \left( \partial_\mu \varphi_1 - \frac{1}{2} i A_\mu^- \varphi_2 - \frac{1}{2} i A_\mu^3 \varphi_1 - \frac{1}{2} i \tan \theta B_\mu \varphi_1 \right) x_\alpha \right. \\
& + \left( \partial_\mu \varphi_2 - \frac{1}{2} i A_\mu^+ \varphi_1 + \frac{1}{2} i A_\mu^+ \varphi_2 - \frac{1}{2} i B_\mu \varphi_2 \tan \theta \right) x_\beta \\
& - \left( \partial_\mu \varphi_1^* + \frac{1}{2} i A_\mu^+ \varphi_2^* + \frac{1}{2} i A_\mu^3 \varphi_1^* + \frac{1}{2} i B_\mu \varphi_1^* \tan \theta \right) x_{-\alpha} \\
& \left. - \left( \partial_\mu \varphi_2^* + \frac{1}{2} i A_\mu^- \varphi_1^* - \frac{1}{2} i A_\mu^3 \varphi_2^* + \frac{1}{2} i \tan \theta B_\mu \varphi_2^* \right) x_{-\beta} \right]
\end{aligned} \tag{3.87}$$

where

$$e^* \omega_E = A_\mu^i \bar{\theta}^\mu t_i + B_\mu \bar{\theta}^\mu y \tag{3.88}$$

$$e^* \omega = \alpha_i^c A_\mu^i \bar{\theta}^\mu \tilde{t}_i + \Phi_a^a \theta^{\tilde{a}} X_a, \tag{3.89}$$

$$\tilde{t}_i = t_i, \quad i = 1, 2, 3, \quad \tilde{t}_4 = y. \tag{3.90}$$

Let us proceed a spontaneous symmetry breaking and Higgs' mechanism. In this way we transform

$$\nabla_\mu^{\text{gauge}} \Phi_{\tilde{a}} \mapsto \text{ad}'_{U^{-1}(x)} \nabla_\mu^{\text{gauge}} \Phi_{\tilde{a}} = \nabla_\mu^{\text{gauge}} \Phi_{\tilde{a}}^u, \quad \tilde{a} = 5, 6, \tag{3.91}$$

where

$$\begin{aligned}
\nabla_\mu^{\text{gauge}} \Phi_5^u &= \frac{1}{2\sqrt{2}} \left[ \partial_\mu H(x) (x_\beta - x_{-\beta}) \right. \\
& \left. + \frac{i}{2} (v + H(x)) (A_\mu^{3u} (x_\beta + x_{-\beta}) + B_\mu \tan \theta (x_{-\beta} - x_\beta) - A_\mu^{+u} x_{-\alpha} + A_\mu^{-u} x_\alpha) \right]
\end{aligned} \tag{3.92}$$

$$\begin{aligned}
\nabla_\mu^{\text{gauge}} \Phi_6^u &= \frac{\sin \psi}{2i} \left[ \partial_\mu H(x) (x_\beta + x_{-\beta}) \right. \\
& \left. + \frac{i}{2} (v + H(x)) (A_\mu^{+u} x_{-\alpha} - A_\mu^{-u} x_\alpha + A_\mu^{3u} (x_\beta - x_{-\beta}) + B_\mu \tan \theta (x_{-\beta} - x_\beta)) \right]
\end{aligned} \tag{3.93}$$

where

$$H_{56}^u = -\frac{\sin \psi (v + H(x))}{2} \left( (v + H(x)) \frac{\beta_i}{\beta \cdot \beta} H_i + \sqrt{2} \cos \psi (x_\beta + x_{-\beta}) \right) \tag{3.94}$$

$$H_{56}^u = -H_{65}^u \tag{3.95}$$

where

$$A_\mu^+ \mapsto A_\mu^{+u} = (\text{ad}'_{U^{-1}(x)} A_\mu)^+ + \frac{i}{2v} \partial_\mu t^+(x) \tag{3.96}$$

$$A_\mu^- \mapsto A_\mu^{-u} = (\text{ad}'_{U^{-1}(x)} A_\mu)^- + \frac{i}{2v} \partial_\mu t^-(x) \tag{3.97}$$

$$A_\mu^3 \mapsto A_\mu^{3u} = (\text{ad}'_{U^{-1}(x)} A_\mu)^3 + \frac{i}{2v} \partial_\mu t^3(x). \quad (3.98)$$

Let us suppose that  $H = G2$ . In this case one gets

$$\begin{aligned} |\beta| &= |\alpha| = \sqrt{2}, \quad |\gamma| = \sqrt{6}, \\ \alpha \cdot \alpha &= \beta \cdot \beta = 2, \quad \gamma \cdot \gamma = 6, \\ \langle \gamma, \alpha \rangle &= 3, \quad \langle \gamma, \beta \rangle = \langle \alpha, \beta \rangle = -1, \\ \frac{\gamma_1 \alpha_2 - \gamma_2 \alpha_1}{\gamma \cdot \gamma} &= \frac{\sqrt{3}}{6}, \\ \theta &= 30^\circ, \quad \cos \theta = \frac{\sqrt{3}}{2}, \quad \sin \theta = \frac{1}{2}. \end{aligned} \quad (3.99)$$

## 4 Spinor fields on $P$

Let  $\Psi$  be a spinor field on  $P$  belonging to fundamental representation  $D^F$  of  $\text{SO}(1, n+3)$  ( $\text{Spin}(1, n+3)$ ) or  $\text{SO}(1, n+n_1+3)$  ( $\text{Spin}(1, n+n_1+3)$ ) or also  $\text{SO}(1, 19)$  ( $\text{Spin}(1, 19)$ ) and  $\Gamma^A$ ,  $A = 1, 2, \dots, n+4$  be a representation of the Clifford algebra for  $\text{SO}(1, n+3)$  acting in the space representation  $D^F$  (see Refs [54], [55], [56]), i.e.,  $\Gamma^A \in C(1, n+3)$ ,

$$\begin{aligned} \{\Gamma^A, \Gamma^B\} &= 2\eta_{AB}, \\ \Gamma^A &\in L(\mathbb{C}^k), \quad k = 4 \cdot 2^{[n/2]}, \quad [n/2] = l, \\ \eta_{AB} &= \text{diag}(-1, -1, -1, 1, \underbrace{-1, -1, \dots, -1}_{n \text{ times}}). \end{aligned} \quad (4.1)$$

We consider also two additional cases mentioned above:

$$\{\Gamma^{\tilde{A}}, \Gamma^{\tilde{B}}\} = 2\eta_{\tilde{A}\tilde{B}}, \quad \Gamma^{\tilde{A}} \in L(\mathbb{C}^k),$$

where

$$k = 4 \cdot 2^{[(n+n_1)/2]}, \quad \eta_{\tilde{A}\tilde{B}} = (-1, -1, -1, 1, \underbrace{-1, -1, \dots, -1}_{n+n_1 \text{ times}}).$$

We introduce a spinor field  $\overline{\Psi}$

$$\overline{\Psi} = \Psi^+ B \quad (4.2)$$

where “+” is a Hermitian conjugation and

$$\Gamma^{A+} = B \Gamma^A B^{-1}. \quad (4.3)$$

Usually  $B = \Gamma^4$ .

It is easy to see that

$$\overline{\Psi}(pg_1) = \overline{\Psi}(p)\sigma(g_1), \quad (4.4)$$

$p = (x, g) \in \underline{P}$ ,  $g, g_1 \in G$  (or  $H$ ),  $\sigma$  is a unitary representation of the group  $G$  (or  $H$ ) acting in  $k = 4 \cdot 2^l$ -dimensional complex space  $\sigma \in L(\mathbb{C}^k)$ . The fields  $\Psi$  and  $\overline{\Psi}$  are defined on  $\underline{P}$  and

$P$  is assumed to have an orthonormal coordinate system  $\theta^A$  (or  $\theta^{\tilde{A}}$ ). This coordinate system is in general non-holonomic. We perform an infinitesimal change of the frame  $\theta^A$  (or  $\theta^{\tilde{A}}$ )

$$\theta^{A'} = \theta^A + \delta\theta^A = \theta^A - \varepsilon^A_{AB}\theta^B, \quad \varepsilon_{AB} + \varepsilon_{BA} = 0 \quad (4.5)$$

or

$$\theta^{\tilde{A}'} = \theta^{\tilde{A}} + \delta\theta^{\tilde{A}} = \theta^{\tilde{A}} - \varepsilon^{\tilde{A}}_{\tilde{A}\tilde{B}}\theta^{\tilde{B}}, \quad \varepsilon_{\tilde{A}\tilde{B}} + \varepsilon_{\tilde{B}\tilde{A}} = 0. \quad (4.6)$$

Suppose that the field  $\Psi$  corresponds to  $\theta^A$  (or  $\theta^{\tilde{A}}$ ) and  $\Psi'$  to  $\theta^{A'}$  ( $\theta^{\tilde{A}'}$ ), then we get

$$\begin{aligned} \Psi' &= \Psi + \delta\Psi = \Psi - \varepsilon_{AB}\hat{\sigma}^{AB}\Psi, \\ \bar{\Psi}' &= \bar{\Psi} + \delta\bar{\Psi} = \bar{\Psi} + \bar{\Psi}\varepsilon_{AB}\hat{\sigma}^{AB}, \end{aligned} \quad (4.7)$$

where

$$\hat{\sigma}^{AB} = \frac{1}{8}[\Gamma^A, \Gamma^B].$$

Simultaneously we have

$$\begin{aligned} \Psi' &= \Psi + \delta\Psi = \Psi - \varepsilon_{\tilde{A}\tilde{B}}\hat{\sigma}^{\tilde{A}\tilde{B}}\Psi, \\ \bar{\Psi}' &= \bar{\Psi} + \delta\bar{\Psi} = \bar{\Psi} + \bar{\Psi}\varepsilon_{\tilde{A}\tilde{B}}\hat{\sigma}^{\tilde{A}\tilde{B}}, \\ \hat{\sigma}^{\tilde{A}\tilde{B}} &= \frac{1}{8}[\Gamma^{\tilde{A}}, \Gamma^{\tilde{B}}]. \end{aligned} \quad (4.8)$$

Now we consider covariant derivatives of spinor fields  $\Psi$  and  $\bar{\Psi}$  on  $P$  with respect to a connection  $\tilde{\omega}^A_B$  (or  $\tilde{\omega}^{\tilde{A}}_{\tilde{B}}$ ) generated by a symmetric metric tensor  $\gamma_{(AB)}$  (or  $\gamma_{(\tilde{A}\tilde{B})}$ ). Both connections are Levi-Civita connections. We get

$$\begin{aligned} D\Psi &= d\Psi + \omega^{AB}\hat{\sigma}_{AB}\Psi \\ D\bar{\Psi} &= d\bar{\Psi} - \bar{\Psi}\omega^{AB}\hat{\sigma}_{AB} \end{aligned} \quad (4.9)$$

or

$$\begin{aligned} D\Psi &= d\Psi + \omega^{\tilde{A}\tilde{B}}\hat{\sigma}_{\tilde{A}\tilde{B}}\Psi \\ D\bar{\Psi} &= d\bar{\Psi} - \bar{\Psi}\omega^{\tilde{A}\tilde{B}}\hat{\sigma}_{\tilde{A}\tilde{B}} \end{aligned} \quad (4.10)$$

Moreover, we consider in Refs [17], [18] new kinds of “gauge” derivatives. We generalized this approach to an arbitrary gauge group  $G$  (or  $H$ ) (see Ref. [19])

$$\begin{aligned} \mathcal{D}\Psi &= \text{hor}(D\Psi) = \overset{\text{gauge}}{d}\Psi + \text{hor}(\omega^{AB})\hat{\sigma}_{AB}\Psi \\ \mathcal{D}\bar{\Psi} &= \text{hor}(D\bar{\Psi}) = \overset{\text{gauge}}{d}\bar{\Psi} + \bar{\Psi}\text{hor}(\omega^{AB})\hat{\sigma}_{AB} \end{aligned} \quad (4.11)$$

or

$$\begin{aligned} \mathcal{D}\Psi &= \text{hor}(D\Psi) = \overset{\text{gauge}}{d}\Psi + \text{hor}(\omega^{\tilde{A}\tilde{B}})\hat{\sigma}_{\tilde{A}\tilde{B}}\Psi \\ \mathcal{D}\bar{\Psi} &= \text{hor}(D\bar{\Psi}) = \overset{\text{gauge}}{d}\bar{\Psi} + \bar{\Psi}\text{hor}(\omega^{\tilde{A}\tilde{B}})\hat{\sigma}_{\tilde{A}\tilde{B}} \end{aligned} \quad (4.12)$$

Horizontality is understood in the sense of a connection  $\omega$  on the bundle  $\underline{P}$ .

It is easy to see that a connection

$$\hat{\omega}_{AB} = \text{hor}(\tilde{\omega}_{AB}) \quad (4.13)$$

defined by our new gauge derivatives is a metric connection on  $P$ , moreover, with a non-vanishing torsion. In this way we get a consistent theory of covariant differentiation of spinor and tensor (vector) fields working with this connection in place of  $\tilde{\omega}_{AB}$ . This remark is also applied for the connection

$$\hat{\omega}_{\tilde{A}\tilde{B}} = \text{hor}(\tilde{\omega}_{\tilde{A}\tilde{B}}) \quad (4.14)$$

We refer also to Appendix A and Appendix B.

Let us notice the following fact. All the formalism considered here can be extended to Rarita–Schwinger field, i.e. to 3/2-spin (not only to 1/2-spin) field. In order to do this we consider one-form spinor fields. It means, we consider on  $P$  horizontal 1-forms (horizontality is understood in the sense of a connection  $\omega$  on a principal fiber bundle  $\underline{P}$ ). They take values in a fundamental representation of  $\text{SO}(1, n+3)$  ( $\text{Spin}(1, n+3)$ ) or  $\text{SO}(1, n+n_1+3)$  ( $\text{Spin}(1, n+n_1+3)$ ) or  $\text{SO}(1, 19)$  ( $\text{Spin}(1, 19)$ ). We have

$$\Psi = \Psi_M \theta^M, \quad \bar{\Psi} = \bar{\Psi}_M \theta^M, \quad \bar{\Psi}_M = \Psi_M^+ \Gamma^4 \quad (4.15)$$

$$\Psi = \Psi_\mu \theta^\mu, \quad \bar{\Psi} = \bar{\Psi}_\mu \theta^\mu, \quad \bar{\Psi}_\mu = \Psi_\mu^+ \Gamma^4. \quad (4.16)$$

Eqs (4.15) correspond to the case with a spontaneous symmetry breaking, and Eqs (4.16) to the case without a spontaneous symmetry breaking (an absence of a manifold  $M = G/G_0$  or  $S^2$ , see Ref. [20]).

We assume that these one-form spinor fields depend on the group coordinates in a trivial way, i.e. by the action of the group  $G$  ( $H$  or  $G_2$ ). We introduce for these one-form spinor fields a new kind of gauge derivatives similarly as for a 0-form spinor fields case

$$\mathcal{D}\Psi = \text{hor}(D\Psi) = d_1\Psi + \text{hor}(\tilde{\omega}^{AB})\hat{\sigma}_{AB} \wedge \Psi \quad (4.17)$$

$$\mathcal{D}\bar{\Psi} = \text{hor}(D\bar{\Psi}) = d_1\bar{\Psi} - \text{hor}(\tilde{\omega}^{AB}) \wedge \bar{\Psi}\hat{\sigma}_{AB} \quad (4.18)$$

where

$$D\Psi = d\Psi + \tilde{\omega}^{AB}\hat{\sigma}_{AB} \wedge \Psi \quad (4.19)$$

$$D\bar{\Psi} = d\bar{\Psi} - \tilde{\omega}^{AB} \wedge \bar{\Psi}\hat{\sigma}_{AB} \quad (4.20)$$

$$d_1\Psi = \text{hor}(d\Psi) \quad (4.21)$$

$$d_1\bar{\Psi} = \text{hor}(d\bar{\Psi}) \quad (4.22)$$

are ordinary “gauge” derivatives on  $E$  (or on  $E \times M$ ) if we take in the place of  $A, B$  also  $\tilde{A}, \tilde{B}$ . In this way we take similarly

$$\mathcal{D}\Psi = d_1\Psi + \text{hor}(\tilde{\omega}^{\tilde{A}\tilde{B}})\hat{\sigma}_{\tilde{A}\tilde{B}} \wedge \Psi \quad (4.23)$$

$$\mathcal{D}\bar{\Psi} = d_1\bar{\Psi} - \text{hor}(\tilde{\omega}^{\tilde{A}\tilde{B}}) \wedge \bar{\Psi}\hat{\sigma}_{\tilde{A}\tilde{B}} \quad (4.24)$$

It is understandable that we have in both cases Eq. (4.4) for 1-form spinor fields.

It is easy to see that we get

$$\mathcal{D}\Psi = \overset{\text{gauge}}{\tilde{\mathcal{D}}}\Psi - \frac{1}{8}\lambda H_\gamma^{a\ b}[\Gamma_a, \Gamma_b]\theta^\gamma \wedge \Psi \quad (4.25)$$

$$\mathcal{D}\bar{\Psi} = \overset{\text{gauge}}{\tilde{\mathcal{D}}}\bar{\Psi} + \frac{1}{8}\lambda H_\gamma^{a\ b}\theta^\gamma \wedge \bar{\Psi}[\Gamma_a, \Gamma_b], \quad (4.26)$$

where

$$\overset{\text{gauge}}{\tilde{\mathcal{D}}} \Psi = \text{hor}(\tilde{D}\Psi) \quad (4.27)$$

$$\overset{\text{gauge}}{\tilde{\mathcal{D}}} \bar{\Psi} = \text{hor}(\tilde{D}\bar{\Psi}). \quad (4.28)$$

$\overset{\text{gauge}}{\tilde{\mathcal{D}}} \Psi$  and  $\overset{\text{gauge}}{\tilde{\mathcal{D}}} \bar{\Psi}$  are exterior covariant derivatives on  $E$  (or  $E \times M = E \times G/G_0$ , or  $E \times S^2$ ) with respect to a Levi-Civita connection on  $E$  (or  $E \times M = E \times G/G_0$ , or  $E \times S^2$ ) and “gauge” at once.

## 5 Lagrangians for fermion fields

Let us define lagrangians for fermion fields defined on a manifold  $P$  in several cases (see Refs [57]–[66]). Let  $\Psi(x, y)$  be a spinor field in a fundamental representation of  $\text{SO}(1, n+3)$  ( $\text{Spin}(1, n+3)$ ) where  $x \in E$ ,  $y \in G$  (in a local trivialization). Let  $\Phi_i(y)$  be zero modes on  $G$ , i.e.

$$\overset{(\text{int})}{\mathbb{D}} \Phi_i = 0 \quad (5.1)$$

where

$$\overset{(\text{int})}{\mathbb{D}} = \Gamma^a \partial_a, \quad a = 5, 6, \dots, n+4. \quad (5.2)$$

In this way we can write

$$\Psi(x, y) = \sum_i \Psi_i(x) \Phi_i(y) \quad (5.3)$$

and

$$\overset{(\text{int})}{\mathbb{D}} \Psi = 0. \quad (5.4)$$

It means we consider on  $P$  only zero-modes with respect to  $G$

$$\bar{\Psi}(x, y) = \Psi^+(x, y) \Gamma^4 = \sum_i \Psi_i^+(x) \Phi_i^*(y) \Gamma^4 = \sum_i \bar{\Psi}_i(x) \Phi_i^*(y). \quad (5.5)$$

Our zero-modes on  $G$  form a complete orthonormal basis

$$\int_G d\mu(y) \Phi_i(y) \Phi_j^*(y) = \delta_{ij}. \quad (5.6)$$

In this way we define a lagrangian for fermions

$$\mathcal{L}_{\text{fermions}} = \frac{1}{2} i\hbar c \int_G d\mu_G(y) \left( \bar{\Psi} \Gamma^M \overset{\text{gauge}}{\tilde{\nabla}}_M \Psi - \overset{\text{gauge}}{\tilde{\nabla}}_M \bar{\Psi} \Gamma^M \Psi \right). \quad (5.7)$$

$\mu_G$  is a biinvariant measure on  $G$ .

Moreover, our covariant derivative (defined in Section 4) is with respect to a horizontal part of Levi-Civita connection on  $P$  and one gets (see Appendix A and B)

$$\begin{aligned}
\mathcal{L}_{\text{fermions}} &= \frac{1}{2} i\hbar c \sum_i \left( \bar{\Psi}_i(x^\alpha) \Gamma^\mu \overset{\text{gauge}}{\widetilde{\nabla}}_\mu \Psi_i(x^\alpha) - \overset{\text{gauge}}{\widetilde{\nabla}}_\mu \bar{\Psi}_i(x^\alpha) \Gamma^\mu \Psi_i(x^\alpha) \right) \\
&= \sum_i \mathcal{L}_D(\Psi_i, \bar{\Psi}_i, \overset{\text{gauge}}{\widetilde{D}}) + i \frac{4\ell_{\text{pl}}}{\sqrt{\alpha}} q H^{a\alpha\gamma} h_{ab} \bar{\Psi}_i \Gamma^b [\Gamma_\alpha, \Gamma_\gamma] \Psi_i, \quad (5.8)
\end{aligned}$$

$\mathcal{L}_D(\Psi_i, \bar{\Psi}_i, \overset{\text{gauge}}{\widetilde{\nabla}})$  is an ordinary Dirac lagrangian for spinor field  $\Psi_i$ , where  $\overset{\text{gauge}}{\widetilde{D}}$  means an ordinary exterior covariant derivative with respect to Levi-Civita connection on  $E$  and “gauge” at once. We use of course Eq. (5.6).  $\ell_{\text{pl}}$  means a Planck’s length,  $q$ —an elementary charge,  $\alpha$  a fine structure constant. We get here some anomalous terms which can go to PC breaking (see Ref. [22]). Let us come to more complicated case where we have to do with a spontaneous symmetry breaking and Higgs’ field, i.e. we define spinor fields on  $P$  which is now a bundle manifold over  $E \times G/G_0$  with a structural group  $H$ . In this case we have

$$\Psi(x, x_1, y) = \sum_i \sum_k \Psi_{ik}(x^\alpha) \tilde{\Phi}_i(x_1) \hat{\Phi}(y), \quad x \in E, \quad x_1 \in M = G/G_0, \quad y \in H, \quad (5.9)$$

(in a local trivialization), where  $\hat{\Phi}^k(y)$  are zero-modes on  $H$ ,

$$\int d\mu_H(y) \hat{\Phi}_k(y) \hat{\Phi}_l(y) = \delta_{kl} \quad (5.10)$$

$$\overset{(\text{int})}{\mathbb{D}} \hat{\Phi}_k = 0. \quad (5.11)$$

$\hat{\Phi}_i(x_1)$ ,  $x_1 \in M = G/G_0$  are not in general zero-modes on  $M$ . Moreover, these functions form an orthonormal set defined on  $M$ ,

$$\int_M d\mu_M(x_1) \tilde{\Phi}_1(x_1) \hat{\Phi}_j^*(x_1) = \delta_{ij}, \quad (5.12)$$

$d\mu_H(y)$  and  $d\mu_M(x_1)$  are measures on  $H$  and  $M$ , respectively.  $r$  is a radius of the manifold  $M$  which gives us a scale of masses of fermions. Usually it is supposed that  $\mu_H$  is a biinvariant measure on  $H$ .

$$\bar{\Psi} = \Gamma^4 \Psi^+ = \sum_i \sum_k \bar{\Psi}_{ik}(x^\alpha) \tilde{\Phi}^*(x_1) \hat{\Phi}_k^*(y) \quad (5.13)$$

$$\bar{\Psi}_{ik} = \Gamma^4 \Psi_{ik}^+. \quad (5.14)$$

The lagrangian for fermions is defined in this case as follows

$$\mathcal{L}_{\text{fermions}} = \frac{i\hbar c}{2} \int_M d\mu_M(x_1) \int d\mu_H(y) \left( \bar{\Psi} \Gamma^M \overset{\text{gauge}}{\widetilde{\nabla}}_M \Psi - \overset{\text{gauge}}{\widetilde{\nabla}}_M \bar{\Psi} \Gamma^M \Psi \right). \quad (5.15)$$

One easily gets

$$\mathcal{L}_{\text{fermions}} = \sum_i \sum_k \left( \mathcal{L}_D(\Psi_{ik}, \bar{\Psi}_{ik}, \overset{\text{gauge}}{\widetilde{D}}) + i \frac{4\ell_{\text{pl}} q}{\sqrt{\alpha}} H^{a\alpha\gamma} h_{ab} \bar{\Psi}_{ik} \Gamma^b [\Gamma_\alpha, \Gamma_\gamma] \Psi_{ik} \right)$$

$$\begin{aligned}
& + \frac{i\hbar c}{r} \Phi_{\tilde{m}}^a \bar{\Psi}_{ik} \Gamma^{\tilde{m}} \hat{X}_a \Psi_{ik} - \frac{\hbar c}{8r} \nabla_{\beta}^{\text{gauge}} \bar{\Phi}^a \bar{\Psi}_{ik} \Gamma^{\tilde{m}} [\Gamma^b, \Gamma^b] h_{ab} \Psi_{ik} \\
& + \frac{i\hbar c}{2r} \sum_j \bar{\Psi}_{ij} \Gamma^{\tilde{m}} \hat{\rho}_{\tilde{m}kj} \Psi_{ik} \quad (5.16)
\end{aligned}$$

where

$$\hat{\rho}_{\tilde{m}kj} = \int_M d\tilde{\mu}_M(x_1) (\tilde{\Phi}_j^*(x_1) \xi_{\tilde{m}} \tilde{\Phi}_k(x_1) + \xi_{\tilde{m}} \tilde{\Phi}_k^*(x_1) \tilde{\Phi}_j(x_1)), \quad (5.17)$$

$\tilde{\mu}_M(x_1)$  is normalized measure on  $M$ . It means we take  $g_{(\tilde{a}\tilde{b})}$  (see Eq. (3.17)).

$\hat{X}_a$  mean matrices of  $\mathfrak{h}$  algebra generators in a spinor representation,  $\xi_{\tilde{a}}$  vector fields acting along coordinate lines on  $M = G/G_0$ ,  $\mathcal{L}_D(\Psi_{ik}, \bar{\Psi}_{ik}, \overset{\text{gauge}}{\tilde{D}})$  is an ordinary Dirac lagrangian for  $\Psi_{ik}$ .

$\overset{\text{gauge}}{\tilde{D}}$  means an exterior covariant derivative with respect to a Levi-Civita connection generated by  $g_{(\alpha\beta)}$  on  $E$  and “gauge” at once.

It is easy to see that

$$i\hbar c \sum_i \sum_k \Phi_{\tilde{m}}^a \bar{\Psi}_{ik} \Gamma^{\tilde{m}} \hat{X}_a \Psi_{ik} \quad (5.18)$$

is a Yukawa interaction term and can be a source of masses for fermions after a spontaneous symmetry breaking and Higgs’ mechanism.

The term

$$\frac{i\hbar c}{2r} \sum_j \bar{\Psi}_{ij} \Gamma^{\tilde{m}} \hat{\rho}_{\tilde{m}kj} \Psi_{ik} \quad (5.19)$$

can be also a source of masses and mixing angles for fermions. Moreover, the term

$$i \frac{4\ell_{\text{pl}} q}{\sqrt{\alpha}} H^{a\alpha\gamma} h_{ab} \bar{\Psi}_{ik} \Gamma^b [\Gamma_{\alpha}, \Gamma_{\gamma}] \Psi_{ik} \quad (5.20)$$

defines an anomalous electric-like dipole interaction with Yang–Mills’ field. We get here anomalous terms which can go to PC breaking (see Ref. [22]).

Let us come to the GSW model in our setting. In this case a lagrangian for fermions is defined:

$$\begin{aligned}
\mathcal{L}_{\text{fermions}} &= \frac{i\hbar c}{2} \int_{S^2 \times G_2} \sin \psi d\psi d\varphi d\mu_{G_2}(y) (\bar{\Psi} \Gamma^M \overset{\text{gauge}}{\tilde{\nabla}}_M \Psi - \overset{\text{gauge}}{\tilde{\nabla}}_M \bar{\Psi} \Gamma^M \Psi) \\
&= \frac{i\hbar c}{2} \sum_i \int_{S^2} \sin \psi d\psi d\varphi \left( \sum_{\substack{j,m \\ |m| \leq j}} \bar{\Psi}_{ijm} Y_{jm}^*(\psi, \varphi) \Gamma^M \overset{\text{gauge}}{\tilde{\nabla}}_M \sum_{\substack{l,n \\ |n| \leq l}} \bar{\Psi}_{iln} Y_{ln}(\psi, \varphi) \right. \\
&\quad \left. - \overset{\text{gauge}}{\tilde{\nabla}}_M \sum_{\substack{j,m \\ |m| \leq j}} \bar{\Psi}_{ijm} Y_{jm}^*(\psi, \varphi) \Gamma^M \sum_{\substack{l,n \\ |n| \leq l}} \bar{\Psi}_{iln} Y_{ln}(\psi, \varphi) \right) \quad (5.21)
\end{aligned}$$

where  $Y_{ln}$  are spherical harmonics (see Refs [67]–[69]) and

$$\Psi(x, \psi, \varphi, y) = \sum_i \sum_{j,m} \Psi_{ijm}(x) Y_{jm}(\psi, \varphi) \Phi_i(y) \quad (5.22)$$

$$\int_0^\pi \int_0^{2\pi} \sin \psi d\psi d\varphi Y_{jm}(\psi, \varphi) Y_{ln}^*(\psi, \varphi) = \delta_{jl} \delta_{mn} \quad (5.23)$$

$$Y_{jm}(\psi, \varphi) = e^{im\varphi} \sqrt{\frac{(2j+1)(j-m)!}{4\pi(j+m)!}} P_l^m(\cos \psi), \quad (5.24)$$

$j = 0, 1, 2, \dots, m = -j, -j+1, \dots, j-1, j, |m| \leq j$ .  $\Phi_i(y)$  are zero-modes on  $G2$  manifold.

$$\bar{\Psi}(x^\alpha, \psi, \varphi, y) = \sum_i \sum_{j,m} \bar{\Psi}_{ijm}(x^\alpha) Y_{jm}^*(\psi, \varphi) \Phi_i^*(y) \quad (5.25)$$

$$\int_{G2} d\mu_{G2}(y) \Phi_i(y) \Phi_j^*(y) = \delta_{ij}. \quad (5.26)$$

They form an orthonormal basis on  $G2$ .

$P_l^m(x)$  are associated Legendre polynomials defined on  $\langle -1, 1 \rangle$  intervals for  $|m| \leq l$ . In this way we can proceed calculations:

$$\begin{aligned} \mathcal{L}_{\text{fermions}} = & \frac{i\hbar c}{2} \sum_i \left\{ \sum_{j,m} \sum_{l,n} \int_{S^2} \sin \psi d\psi d\varphi \left[ (\bar{\Psi}_{ijm} \Gamma^\mu \overset{\text{gauge}}{\tilde{\nabla}}_\mu \Psi_{iln} Y_{jm}^*(\psi, \varphi) Y_{ln}(\psi, \varphi) \right. \right. \\ & - \overset{\text{gauge}}{\tilde{\nabla}}_\mu \bar{\Psi}_{ilm} \Gamma^\mu \Psi_{ijm} Y_{lm}^*(\psi, \varphi) Y_{jn}(\psi, \varphi) + (\bar{\Psi}_{ijm} \Gamma^{\tilde{m}} \Psi_{iln} Y_{jm}^*(\psi, \varphi) \xi_{\tilde{m}} Y_{ln}(\psi, \varphi) \\ & \left. \left. - \bar{\Psi}_{ijm} \Gamma^{\tilde{m}} \Psi_{iln} \xi_{\tilde{m}} Y_{jm}^*(\psi, \varphi) Y_{ln}(\psi, \varphi) \right) \right] \\ & = \frac{i\hbar c}{2} \sum_i \sum_{j,m} \left( \bar{\Psi}_{ijm} \Gamma^\mu \overset{\text{gauge}}{\tilde{\nabla}}_\mu \Psi_{ijm} - \overset{\text{gauge}}{\tilde{\nabla}}_\mu \bar{\Psi}_{ijm} \Gamma^\mu \Psi_{ijm} \right) \\ & + \int_0^\pi \int_0^{2\pi} \sin \psi d\psi d\varphi \sum_{j,m} \sum_{l,n} \left[ \left( \bar{\Psi}_{ijm} \Gamma^5 (\varphi_1^* x_{-\alpha} + \varphi_2^* x_{-\beta} - \varphi_1 x_\alpha - \varphi_2 x_\beta) \Psi_{iln} Y_{jm}^*(\psi, \varphi) \right. \right. \\ & - i \sin \psi \bar{\Psi}_{ijm} \Gamma^6 (\varphi_1 x_\alpha + \varphi_2 x_\beta + \varphi_1^* x_{-\alpha} + \varphi_2^* x_\beta) \Psi_{iln} Y_{jm}^*(\psi, \varphi) Y_{ln}(\psi, \varphi) \\ & - i \cos \psi \bar{\Psi}_{ijm} \Gamma^6 (h_\alpha + h_\beta) \Psi_{iln} Y_{jm}^*(\psi, \varphi) Y_{ln}(\psi, \varphi) \\ & + \frac{i}{4} \left[ \bar{\Psi}_{ijm} \left( \partial_\mu \varphi_1 - \frac{1}{2} i A_\mu^- \varphi_2 - \frac{1}{2} i A_\mu^3 \varphi_1 - \frac{\sqrt{3}}{6} B_\mu \varphi_1 \right) x_\alpha h^{x_\alpha a} \right. \\ & + \left( \partial_\mu \varphi_1 - \frac{1}{2} i A_\mu^+ \varphi_1 - \frac{\sqrt{3}}{6} i B_\mu \varphi_2^* \right) x_{-\alpha} h^{x_{-\alpha} a} + \left( \partial_\mu \varphi_1 - \frac{1}{2} i A_\mu^+ \varphi_1 + \frac{1}{2} i A_\mu^+ \varphi_2 - \frac{\sqrt{3}}{6} i B_\mu \varphi_2 \right) x_\beta h^{x_\beta a} \\ & - \left( \partial_\mu \varphi_2^* + \frac{1}{2} i A_\mu^- \varphi_1^* - \frac{1}{2} A_\mu^3 \varphi_2^* + \frac{\sqrt{3}}{6} i B_\mu \varphi_2^* \right) x_{-\beta} h^{x_{-\beta} a} \left. \right] \Gamma^5 [\Gamma^\mu, \Gamma_a] \Psi_{iln} Y_{jm}^*(\psi, \varphi) Y_{ln}(\psi, \varphi) \\ & - i \sin \psi \bar{\Psi}_{ijm} \left[ \left( \partial_\mu \varphi_1 - \frac{1}{2} i A_\mu^+ \varphi_2 - \frac{1}{2} A_\mu^3 \varphi_1 - \frac{\sqrt{3}}{6} i B_\mu \varphi_1 \right) x_\alpha h^{x_\alpha a} \right. \\ & + \left( \partial_\mu \varphi_2 - \frac{1}{2} i A_\mu^+ \varphi_2 - \frac{\sqrt{3}}{6} i B_\mu \varphi_2 \right) x_\beta h^{x_\beta a} - \left( \partial_\mu \varphi_1^* + \frac{1}{2} i A_\mu^3 \varphi_1^* + \frac{\sqrt{3}}{6} i B_\mu \varphi_1^* \right) x_{-\alpha} h^{x_{-\alpha} a} \\ & \left. - \left( \partial_\mu \varphi_2^* + \frac{1}{2} A_\mu^- \varphi_1^* - \frac{1}{2} i A_\mu^3 \varphi_2^* + i \frac{\sqrt{3}}{6} B_\mu \varphi_2^* \right) x_{-\beta} h^{x_{-\beta} a} \right] \Gamma^6 [\Gamma^\mu, \Gamma^a] \Psi_{iln} Y_{jm}^*(\psi, \varphi) Y_{ln}(\psi, \varphi) \left. \right] \\ & + \left\{ \bar{\Psi}_{ijm} \Gamma_{(in)}^5 \Psi_{iln} Y_{jm}^*(\psi, \varphi) Y_{ln}(\psi, \varphi) + \frac{i}{8} \sin 2\psi \bar{\Psi}_{ijm} (\Gamma^5 + 3\Gamma^6) \Psi_{iln} Y_{jm}^*(\psi, \varphi) Y_{ln}(\psi, \varphi) \right. \\ & \left. + \bar{\Psi}_{ijm} \Gamma^5 \Psi_{iln} Y_{jm}^*(\psi, \varphi) \frac{\partial}{\partial \psi} Y_{ln}(\psi, \varphi) \right\} \Big\}, \quad (5.27) \end{aligned}$$



a continuation of Eq. (5.21).

Let us notice the following facts:

$$\begin{aligned}
& -i \int_0^\pi \sin^2 \psi Y_{jm}^*(\psi, \varphi) Y_{ln}(\psi, \varphi) d\psi d\varphi \\
& = -i \frac{\delta_{mn}}{4\pi} \sqrt{\frac{(2l+1)(2j+1)(l-m)!(j-m)!}{(l+m)!(j+m)!}} \int_{-1}^1 dx (1-x^2)^{1/2} P_l^m(x) P_j^m(x) \quad (5.28)
\end{aligned}$$

$$\begin{aligned}
& -i \int_0^\pi \int_0^{2\pi} \sin \psi \cos \psi Y_{jm}^*(\psi, \varphi) Y_{ln}(\psi, \varphi) d\psi d\varphi \\
& = -i \frac{\delta_{mn}}{4\pi} \sqrt{\frac{(2l+1)(2j+1)(l-m)!(j-m)!}{(l+m)!(j+m)!}} \int_{-1}^1 dx x P_l^m(x) P_j^m(x) \quad (5.29)
\end{aligned}$$

$$\begin{aligned}
& \int_0^\pi \int_0^{2\pi} \sin \psi Y_{jm}^*(\psi, \varphi) \frac{\partial}{\partial \psi} Y_{ln}(\psi, \varphi) d\psi d\varphi \\
& = -\frac{\delta_{mn}}{4\pi} \sqrt{\frac{(2l+1)(2j+1)(l-m)!(j-m)!}{(l+m)!(j+m)!}} \\
& \times \left[ (m+1) \int_{-1}^1 \frac{x P_j^m(x) P_l^m(x)}{\sqrt{1-x^2}} dx + (l-m-1) \int_{-1}^1 \frac{x P_j^m(x) P_l^{m+1}(x)}{\sqrt{1-x^2}} dx \right] \quad (5.30)
\end{aligned}$$

$$\int_0^\pi \int_0^{2\pi} \sin \psi Y_{jm}^*(\psi, \varphi) \frac{\partial}{\partial \psi} Y_{ln}(\psi, \varphi) d\psi d\varphi = in \delta_{jl} \delta_{mn} \quad (5.31)$$

$$\begin{aligned}
& \int_0^\pi \int_0^{2\pi} \sin^2 \psi \cos \psi Y_{jm}^*(\psi, \varphi) Y_{ln}(\psi, \varphi) d\psi d\varphi \\
& = \frac{\delta_{mn}}{4} \sqrt{\frac{(2l+1)(2j+1)(l-m)!(j-m)!}{(l+m)!(j+m)!}} \int_{-1}^1 dx x (1-x^2)^{1/2} P_j^m(x) P_l^m(x) \quad (5.32)
\end{aligned}$$

By using Eqs (5.28)–(5.32) our lagrangian for fermions starts to be more simple

$$\begin{aligned}
\mathcal{L}_{\text{fermions}} &= \frac{i\hbar c}{2} \sum_i \left\{ \sum_{j,m} \left[ (\bar{\Psi}_{ijm} \Gamma^\mu \overset{\text{gauge}}{\nabla}_\mu \Psi_{ijm} - \overset{\text{gauge}}{\nabla}_\mu \bar{\Psi}_{ijm} \Gamma^\mu \Psi_{ijm}) \right. \right. \\
& \quad \left. \left. + \bar{\Psi}_{ijm} \Gamma^5 (\varphi_1^* x_{-\alpha} + \varphi_2^* x_{-\beta} - \varphi_1 x_\alpha - \varphi_2 x_\beta) \Psi_{ijm} \right] \right. \\
& + \sum_{j,l,m} \left\{ -\frac{i}{4\pi} \bar{\Psi}_{ijm} \Gamma^6 (\varphi_1 x_\alpha + \varphi_2 x_\beta + \varphi_1^* x_{-\alpha} + \varphi_2^* x_\beta) \Psi_{ijm} \sqrt{\frac{(2l+1)(2j+1)(l-m)!(j-m)!}{(l+m)!(j+m)!}} \right. \\
& \quad \times \int_{-1}^1 dx (1-x^2)^{1/2} P_l^m(x) P_j^m(x) \\
& \quad \left. - \frac{i}{4\pi} \bar{\Psi}_{ijm} \Gamma^6 (h_\alpha + h_\beta) \Psi_{ijm} \sqrt{\frac{(2l+1)(2j+1)(l-m)!(j-m)!}{(l+m)!(j+m)!}} \int_{-1}^1 dx x P_l^m(x) P_j^m(x) \right\} \\
& \quad \frac{i}{4} \left[ \sum_{j,m} \bar{\Psi}_{ijm} \left( \partial_\mu \varphi_1 - \frac{1}{2} i A_\mu^- \varphi_2 - \frac{1}{2} i A_\mu^3 \varphi_1 - \frac{\sqrt{3}}{6} B_\mu \varphi_1 \right) x_\alpha h^{x_\alpha a} \right. \\
& \quad \left. + \left( \partial_\mu \varphi_1 - \frac{i}{2} A_\mu^+ \varphi_1 - \frac{\sqrt{3}}{6} i B_\mu \varphi_2^* \right) x_{-\alpha} h^{x_{-\alpha} a} - \left( \partial_\mu \varphi^* \varphi_2 + \frac{i}{2} A^- \varphi_1^* - \frac{1}{2} A_\mu^3 \varphi_2^* + \frac{\sqrt{3}}{6} i B_\mu \varphi_2^* \right) x_{-\beta} h^{x_{-\beta} a} \right]
\end{aligned}$$

$$\begin{aligned}
& + \left( \partial_\mu \varphi_1 - \frac{1}{2} i A_\mu^+ \varphi_1 + \frac{1}{2} i A_\mu^+ \varphi_2 - \frac{\sqrt{3}}{6} i B_\mu \varphi_2 \right) x_\beta h^{x_\beta a} \Big] \Gamma^5 [\Gamma^\mu, \Gamma_a] \\
& - \frac{i}{4\pi} \sum_{j,l,m} \left\{ \Psi_{ijm} \left[ \left( \partial_\mu \varphi_1 - \frac{1}{2} A_\mu^- \varphi_2 - \frac{i}{2} A_\mu^3 \varphi_1 - \frac{\sqrt{3}}{6} i B_\mu \varphi_1 \right) x_\alpha h^{x_\alpha a} \right. \right. \\
& + \left( \partial_\mu \varphi_2 - \frac{i}{2} A_\mu^+ \varphi_2 - \frac{\sqrt{3}}{6} B_\mu \varphi_2 \right) x_\beta h^{x_\beta a} - \left( \partial_\mu \varphi_1^* + \frac{1}{2} i A_\mu^3 \varphi_1^* + \frac{\sqrt{3}}{6} i B_\mu \varphi_1^* \right) x_{-\alpha} h^{x_{-\alpha} a} \\
& \left. \left. - \left( \partial_\mu \varphi_2^* + \frac{1}{2} A_\mu^- \varphi_1^* - \frac{1}{2} A_\mu^3 \varphi_2^* + i \frac{\sqrt{3}}{6} B_\mu \varphi_2^* \right) x_{-\beta} h^{x_{-\beta} a} \right] \Psi_{ilm} \sqrt{\frac{(2l+1)(2j+1)(l-m)!(j-m)!}{(l+m)!(j+m)!}} \right. \\
& \quad \times \int_{-1}^1 dx x P_l^m(x) P_j^m(x) \\
& + \frac{1}{4} \bar{\Psi}_{ijm} (\Gamma^5 + 3\Gamma^6) \Psi_{ilm} \sqrt{\frac{(2l+1)(2j+1)(l-m)!(j-m)!}{(l+m)!(j+m)!}} \int_{-1}^1 dx x P_l^m(x) P_j^m(x) \Big\} \\
& + i \sum_{j,m} m \bar{\Psi}_{ijm} \Gamma^5 \Psi_{ijm} - \frac{1}{4\pi} \sum_{j,l,m} \bar{\Psi}_{ijm} \Gamma^5 \Psi_{ilm} \sqrt{\frac{(2l+1)(2j+1)(l-m)!(j-m)!}{(l+m)!(j+m)!}} \\
& \quad \times \left[ (m+1) \int_{-1}^1 \frac{x P_j^m(x) P_l^m(x)}{\sqrt{1-x^2}} dx + (l-m-1) \int_{-1}^1 \frac{P_j^m(x) P_l^{m+1}(x)}{\sqrt{1-x^2}} dx \right] \Big\} \quad (5.33)
\end{aligned}$$

In this lagrangian we have a Yukawa interaction which after a spontaneous symmetry breaking and Higgs' mechanism results in masses generation of fermions and possible mixing. These are the terms:

$$\begin{aligned}
1^\circ & \frac{i\hbar c}{2r} \sum_i \sum_{j,m} \Psi_{ijm} \Gamma^5 (\varphi_1^* x_{-\alpha} + \varphi_2^* x_{-\beta} - \varphi_1 x_\alpha - \varphi_2 x_\beta) \Psi_{ijm} \\
2^\circ & \frac{i\hbar c}{2r} \sum_{i,j,l,m} \left\{ -\frac{1}{4\pi} \bar{\Psi}_{ijm} \Gamma^6 (\varphi_1 x_\alpha + \varphi_2 x_\beta + \varphi_2^* x_{-\alpha} + \varphi_1^* x_{-\beta}) \Psi_{ijm} \right. \\
& \quad \sqrt{\frac{(2l+1)(2j+1)(l-m)!(j-m)!}{(l+m)!(j+m)!}} \int_{-1}^1 dx (1-x^2)^{1/2} P_l^m(x) P_j^m(x) \\
& \quad \left. - \frac{1}{4\pi} \bar{\Psi}_{ijm} \Gamma^6 (h_\alpha + h_\beta) \Psi_{ilm} \sqrt{\frac{(2l+1)(2j+1)(l-m)!(j-m)!}{(l+m)!(j+m)!}} \int_{-1}^1 dx x P_l^m(x) P_j^m(x) \right\}
\end{aligned}$$

We have some additional terms which can generate masses and mixing angles

$$\begin{aligned}
1^\circ & -\frac{\hbar c}{2r} \sum_i \sum_m \Psi_{ijm} \Gamma^5 \Psi_{ijm} \\
2^\circ & -\frac{i\hbar c}{4\pi r} \sum_i \sum_{j,l} \bar{\Psi}_{ijm} \Gamma^5 \Psi_{ilm} \sqrt{\frac{(2l+1)(2j+1)(l-m)!(j-m)!}{(l+m)!(j+m)!}} \\
& \quad \times \left[ (m+1) \int_{-1}^1 \frac{x P_j^m(x) P_l^m(x)}{\sqrt{1-x^2}} dx + (l-m-1) \int_{-1}^1 \frac{P_j^m(x) P_l^{m+1}(x)}{\sqrt{1-x^2}} dx \right] \\
3^\circ & -\frac{\hbar c}{16\pi r^2} \sum_{i,j,m} \bar{\Psi}_{ijm} (\Gamma^5 + 3\Gamma^6) \Psi_{ilm} \sqrt{\frac{(2l+1)(2j+1)(l-m)!(j-m)!}{(l+m)!(j+m)!}} \int_{-1}^1 dx P_l^m(x) P_j^m(x),
\end{aligned}$$

$r$  is a radius of a sphere  $S^2$  which gives a scale of masses for fermions.

Thus we have many possibilities in a fermion sector, even if we remove from the theory fermion with Planck's masses via zero-mode condition. In all the formulas written above

$$|m| \leq \min(j, l).$$

The important problem in the theory of fermions is to get chiral fermions. We are supposing of course that the 6-dimensional (in general  $(n_1 + 4)$ -dimensional) mass is zero, i.e.

$$\overset{(\text{int})}{\not{D}} \Psi = 0 \quad (5.34)$$

(i.e. Eq. (5.2)). Otherwise the mass of a fermion is of Planck's mass

$$\overset{(\text{int})}{\not{D}} \Psi = \overline{m} \Psi. \quad (5.35)$$

$\overline{m} \neq 0$  is of order of Planck's mass (see Refs [57]–[66]). Thus we have

$$\overset{(\text{int})}{\not{D}} = \sum_{A=7}^{20} \Gamma^A \partial_A.$$

We demand zero-modes condition. One can write

$$\widehat{I}^5 = \gamma^5 \otimes \Gamma^1, \quad \widehat{I}^6 = \gamma^5 \otimes \Gamma^2, \quad (5.36)$$

$$\not{D}^4 = (\gamma^\mu \otimes I) \partial_\mu, \quad (5.37)$$

$$\overset{(5,6)}{\not{D}} = \gamma^5 \otimes \left( \Gamma^1 \frac{\partial}{\partial \psi} + \Gamma^2 \frac{\partial}{\partial \varphi} \right) \quad (5.38)$$

and finally

$$\overset{(\text{int})}{\not{D}} \Psi = \gamma^5 \otimes \sum_{r=3}^{16} \Gamma^r \partial_{r+4} \Psi. \quad (5.39)$$

If

$$\overset{(\text{int})}{\not{D}} \Phi_i(y) = 0, \quad (5.40)$$

$$\Psi = \sum_i \Psi_i(x^\mu, \psi, \varphi) \Phi_i(y) \quad (5.41)$$

then

$$N_c(\overset{(\text{int})}{\not{D}}) = (n_c^+ - n_c^- - n_{\bar{c}}^+ + n_{\bar{c}}^-) f. \quad (5.42)$$

$n_c^+$  is a number of zero-modes of  $\overset{(\text{int})}{\not{D}}$  for the Weyl spinors  $\Psi^+$  associate with a complex representation of the spinor, while  $n_c^-$  and  $n_{\bar{c}}^+$  denote the corresponding values for  $\Psi^-$  and for the complex conjugate representation.  $\Psi^+$  and  $\Psi^-$  Weyl spinors are defined as usual

$$\begin{aligned} \Psi^\pm &= \frac{1}{2}(1 \pm \overline{I}^{n+5})\Psi, \\ \Psi^\pm &= \frac{1}{2}(1 \pm \overline{I}^{n+n_1+5})\Psi, \\ \Psi^\pm &= \frac{1}{2}(1 \pm \overline{I}^{21})\Psi, \end{aligned}$$

where  $\overline{I}^{n+5}$ ,  $\overline{I}^{n+n_1+5}$ ,  $\overline{I}^{21}$  are higher dimensional analogues of  $\gamma^5$  matrix in four dimensions (see Appendix A). It is understandable that we get left-handed spinors on  $E$  from the expansion  $\Psi^+$  and right-handed from the expansion of  $\Psi^-$ . It means that we get  $\Psi_i^+$  and  $\Psi_i^-$ ,  $\Psi_{ik}^+$ ,  $\Psi_{ik}^-$  or  $\Psi_{ilm}^+$ ,  $\Psi_{ilm}^-$ . The most important case because of physical applications is of course GSW model.

It means  $\Psi_{ilm}^+$ ,  $\Psi_{ilm}^-$  and  $I^{21}$  matrix. Thus  $N_C(\overset{\text{(int)}}{\mathbb{P}})$  is a number of 6-dimensional left-handed fermion generations up to numerical factor, i.e.  $f$  (see Appendix C). Let  $d_C$  be a dimension of a complex representation  $C$ .

We define

$$N = \sum_C d_C |N_C(\overset{\text{(int)}}{\mathbb{P}})|. \quad (5.43)$$

We see that the total number of massless fermions on  $G2$  is

$$n_0(G2) \geq N. \quad (5.44)$$

Let us consider the following problem. Our fermions are labeled by several indices, i.e.  $i$ ,  $ik$ ,  $ilm$  and so on. Let us consider them as labeled by one index  $I$ , i.e.  $\Psi_I$ , in particular  $I = i$ ,  $I = (i, k)$ ,  $I = (i, l, m)$ . Our spinors form of course a tower in many cases, this is an infinite tower. Moreover,  $\Psi_I$  belong to the fundamental representation of a group  $\text{SO}(1, n+3)$  ( $\text{Spin}(1, n+3)$ ),  $\text{SO}(1, n+n_1+3)$  ( $\text{Spin}(1, n+n_1+3)$ ) or in the case of GSW to  $\text{SO}(1, 19)$  ( $\text{Spin}(1, 19)$ ).

We should decompose this representation to a representation of Lorentz group  $\text{SO}(1, 3)$  ( $\text{SL}(2, \mathbb{C}) = \text{Spin}(1, 3)$ ). One gets

$$D^F|_{\text{SO}(1,3)}(\Lambda) = L(\Lambda) \oplus \dots \oplus L(\Lambda), \quad \Lambda \in \text{SO}(1, 3) \quad (5.45)$$

$$L(\Lambda) = D^{(1/2,0)}(\Lambda) \oplus D^{(0,1/2)} \quad (5.46)$$

(see Ref. [56]).

$$\Psi_I|_{\text{SO}(1,3)} = \begin{bmatrix} \Psi_{I1} \\ \vdots \\ \Psi_{I2^{[n/2]}} \end{bmatrix} \quad (5.47)$$

or

$$\Psi_I|_{\text{SO}(1,3)} = \begin{bmatrix} \Psi_{I1} \\ \vdots \\ \Psi_{I2^{[(n+n_1)/2]}} \end{bmatrix} \quad (5.48)$$

$$\Psi_I|_{\text{SO}(1,3)} = \begin{bmatrix} \Psi_{I1} \\ \vdots \\ \Psi_{I2^8} \end{bmatrix} \quad (5.49)$$

After taking a section of a bundle we get in any case

$$e^*(\Psi_I)|_{\text{SO}(1,3)} = \begin{bmatrix} \Psi_{I1} \\ \vdots \\ \Psi_{I2^N} \end{bmatrix} \quad (5.50)$$

where  $N = [\frac{n}{2}]$  or  $[\frac{n+n_1}{2}]$  or 8.

In the case of Weyl spinors we have similarly

$$\begin{aligned}\Psi_{Ji}^+, \quad i = 1, 2, \dots, 2^{N-1} \\ \Psi_{Ji}^-, \quad i = 1, 2, \dots, 2^{N-1},\end{aligned}\tag{5.51}$$

where as usual

$$e^*(\Psi_I^\pm)|_{\text{SO}(1,3)} = \begin{bmatrix} \Psi_{I1}^\pm \\ \vdots \\ \Psi_{I2^{N-1}}^\pm \end{bmatrix}.\tag{5.52}$$

According to Ref. [20] we consider  $\frac{3}{2}$ -spin spinor fields as 1-forms defined on  $P$  horizontal with respect to a connection defined on a fiber bundle  $\underline{P}$ ,

$$\text{hor } \Psi = \Psi.$$

In this way  $\frac{3}{2}$ -spinor (-fields) are represented by spinor-valued forms (-fields). We have three cases as for ordinary  $\frac{1}{2}$ -spinor fields, i.e.  $\underline{P}$  over  $E$  (space-time),  $\underline{P}$  over  $E \times M = E \times G/G_0$  and  $\underline{P}$  over  $E \times S^2$ . Moreover, we develop in details only the first case.

It is easy to see that we can proceed the same expansion procedure for 1-form spinor fields on a manifold  $P$  as for a 0-form spinor fields (i.e. spinor fields). One gets

$$\Psi(x, y) = \sum_i \Psi_i(x) \Phi_i(y)\tag{5.53}$$

where  $\Psi_i(x)$  are now 1-form spinor fields

$$\bar{\Psi}(x, y) = \sum_i \bar{\Psi}_i(x) \bar{\Phi}_i(y), \quad x \in E, \quad y \in G.\tag{5.54}$$

Similarly one finds

$$\Psi(x, x_1, y) = \sum_i \sum_k \Psi_{ik}(x) \tilde{\Psi}_i(x_1) \hat{\Phi}_k(y), \quad x \in E, \quad x_1 \in M = G/G_0, \quad y \in H,\tag{5.55}$$

$$\bar{\Psi}(x, x_1, y) = \sum_i \sum_k \bar{\Psi}_{ik}(x) \tilde{\Psi}_i(x_1) \hat{\Phi}_k(y),\tag{5.56}$$

where as usual

$$\bar{\Psi}_{ik}(x) = \Gamma^4 \Psi_{ik}^+(x)\tag{5.57}$$

$$\bar{\Psi}_i(x) = \Gamma^4 \Psi_i^+(x)\tag{5.58}$$

and also

$$\Psi(x, \psi, \varphi, y) = \sum_i \sum_{\substack{j, m \\ |m| < j}} \Psi_{ijm}(x) Y_{jm}(\psi, \varphi) \Phi_i(y)\tag{5.59}$$

$$\bar{\Psi}(x, \psi, \varphi, y) = \sum_i \sum_{\substack{j, m \\ |m| < j}} \bar{\Psi}_{ijm}(x) Y_{jm}(\psi, \varphi) \Phi_i(y)\tag{5.60}$$

where  $x \in E$ ,  $\psi, \varphi$  are  $S^2$  coordinates,  $y \in G2$  and

$$\bar{\Psi}_{ijm} = \Gamma^4 \Psi_{ijm}^+ \quad (5.61)$$

One can get dimensional reduction procedure similarly as for spinor fields finding

$$e^*(\Psi_I)|_{\text{SO}(1,3)} = \begin{bmatrix} \Psi_{I1} \\ \vdots \\ \Psi_{I2^N} \end{bmatrix} \quad (5.62)$$

where  $N = [\frac{n}{2}]$  or  $[\frac{n+n_1}{2}]$  or 8. But now  $\Psi_{iI}$  are 1-form spinor fields on  $E$  or  $E \times M = E \times G/G_0$ , or  $E \times S^2$ .

One defines the lagrangians for Rarita–Schwinger fields as follows:

$$\mathcal{L}_{\text{Rarita–Schwinger}} = \frac{i\hbar c}{2} \int_M d\mu_M(x_1) \int_H d\mu_H(y) (\bar{\Psi} \wedge \Gamma_{2l+5} \Gamma \wedge \mathcal{D}\Psi - \mathcal{D}\bar{\Psi} \wedge \Gamma_{2l+5} \Gamma \wedge \Psi) \quad (5.63)$$

where

$$\begin{aligned} \Gamma &= \Gamma_M \theta^M \\ \Psi &= \Psi_M \theta^M \\ \bar{\Psi} &= \bar{\Psi}_M \theta^M \end{aligned} \quad (5.64)$$

and we impose supplementary conditions

$$l \wedge \Psi = \bar{\Psi} \wedge l = 0 \quad (5.65)$$

$$l = \Gamma_M \eta^M \quad (5.66)$$

where  $\eta^M$  is a dual Cartan base for  $\theta^M$ . In the case of 20-dimensional model (GSW model) we have  $S^2$  in the place of  $M$ , and  $H = G2$ . In the case without a spontaneous symmetry breaking we have only an integration for  $G$  (which replaces  $H$ ). We will not develop this formalism in the paper. This will be done elsewhere.

In the case without symmetry breaking (an absence of  $M$  or  $S^2$ ) one simply gets in the place of Eqs (5.64)–(5.66)

$$\begin{aligned} \Gamma &= \Gamma_\mu \theta^\mu \\ \Psi &= \Psi_\mu \theta^\mu \\ \bar{\Psi} &= \bar{\Psi}_\mu \theta^\mu \end{aligned} \quad (5.67)$$

$$l \wedge \Psi = \bar{\Psi} \wedge l = 0 \quad (5.68)$$

$$l = \Gamma_\mu \eta^\mu \quad (5.69)$$

where  $\eta^\mu$  is a dual Cartan base for  $\theta^\mu$ .

It is easy to see that in the case with a spontaneous symmetry breaking we have in the theory also a multiplet of spinor 1/2-fields  $\psi_a$ ,  $a = 5, 6, \dots, n_1 + 4$  (or  $a = 5, 6$ ).

In the first case the Rarita–Schwinger field lagrangian looks

$$\mathcal{L}_{\text{Rarita–Schwinger}} = \frac{1}{2} i\hbar c \int_G d\mu_G (\bar{\Psi} \wedge \Gamma_{2l+5} \Gamma \wedge \mathcal{D}\Psi - \mathcal{D}\bar{\Psi} \wedge \Gamma_{2l+5} \Gamma \wedge \Psi)$$

$$\begin{aligned}
&= \sum_j \mathcal{L}_{\text{Rarita-Schwinger}}(\Psi_j, \bar{\Psi}_j, \tilde{\mathcal{D}}) + \frac{i\hbar c\lambda}{2} \sum_j \left[ H^{\alpha\nu b} \Psi_{j\lambda} \Gamma_b [\Gamma_a, \Gamma_\nu] \Psi_j^\lambda \right. \\
&\quad \left. + 2H^{\alpha\rho b} (\bar{\Psi}_{j\lambda} \Gamma_b \Gamma^\lambda \Gamma_\alpha \Psi_{j\rho} + \bar{\Psi}_{j\rho} \Gamma_b \Gamma^\lambda \Gamma_\alpha \Psi_{j\lambda} - \bar{\Psi}_{j\lambda} \Gamma^\lambda \Gamma_\rho \Gamma_b \Gamma^\nu \Gamma_\alpha \Psi_{j\nu}) \right] \eta \quad (5.70)
\end{aligned}$$

where

$$\mathcal{L}_{\text{Rarita-Schwinger}}(\Psi_j, \bar{\Psi}_j, \tilde{\mathcal{D}}) = \frac{i\hbar c}{2} (\bar{\Psi}_j \Gamma_{2l+5} \wedge \Gamma \wedge \tilde{\mathcal{D}} \Psi_j - \tilde{\mathcal{D}} \bar{\Psi}_j \wedge \Gamma_{2l+5} \wedge \Gamma \wedge \Psi_j). \quad (5.71)$$

## Appendix A

In this Appendix we deal with Clifford algebra (see Refs [54], [55])  $C(1, n+3)$  or  $C(1, n+N_1+3)$ . Owing to decomposition rules for  $C(1, n+3)$  ( $C(1, n+N_1+3)$ ) we write down a useful representation for  $\Gamma^A$  (or  $\Gamma^{\tilde{A}}$ ) in terms of  $\gamma^\mu$ . It is well known that any Clifford algebras (see Refs [54], [55]) can be decomposed into a tensor products of the three elementary Clifford algebras

$$\begin{aligned}
C(0, 1) &= C \text{ — complex numbers,} \\
C(1, 0) &= R \oplus R \\
C(0, 2) &= Q \text{ — quaternions.}
\end{aligned} \quad (A.1)$$

We have

$$C(1, n+3) = C(0, 2) \otimes C(1, n+1). \quad (A.2)$$

We define Clifford algebra  $C(1, 3)$  and we easily get

$$C(1, n+3) = \left( \bigotimes_{i=1}^{[n/2]} C(0, 2) \right) \otimes C(1, 3) = \left( \bigotimes_{i=1}^{[n/2]} H \right) \otimes C(1, 3) \quad (A.3)$$

or

$$C(1, n+n_1+3) = \left( \bigotimes_{i=1}^{[(n+n_1)/2]} C(0, 2) \right) \otimes C(1, 3) = \left( \bigotimes_{i=1}^{[(n+n_1)/2]} H \right) \otimes C(1, 3). \quad (A.4)$$

It is very well known that either

$$\begin{aligned}
C(1, n+3) &= C(1, n+4) \quad (C(1, n+n_1+3) = C(1, n+n_1+4)) \\
&\text{iff } n+3 = 2l \quad (n+n_1+3 = 2l)
\end{aligned}$$

or

$$\begin{aligned}
C(1, n+2) &= C(1, n+3) \quad (C(1, n+n_1+2) = C(1, n+n_1+3)) \\
&\text{iff } n+3 = 2l+1 \quad (n+n_1+3 = 2l+1),
\end{aligned}$$

$l \in N_1^\infty$ .

Let  $\gamma^\mu \in L(\mathbb{C}^4)$ ,  $\mu = 1, 2, 3, 4$ , be Dirac matrices obeying conventional relations

$$\begin{aligned}
\{\gamma^\mu, \gamma^\nu\} &= 2\eta^{\mu\nu} \\
\eta^{\mu\nu} &= \text{diag}(-1, -1, -1, -1) \\
\gamma^5 &= \gamma^1 \gamma^2 \gamma^3 \gamma^4, \quad \gamma_5^2 = 1
\end{aligned} \quad (A.5)$$

and let  $\sigma_i \in L(\mathbb{C}^2)$ ,  $i = 1, 2, 3$ , be Pauli's matrices obeying conventional relations as well

$$\begin{aligned}\{\sigma_i, \sigma_j\} &= 2\delta_{ij} \\ [\sigma_i, \sigma_j] &= \varepsilon_{ijk}\sigma_k.\end{aligned}\tag{A.6}$$

Let  $I$  be  $2 \times 2$  unit matrix and let  $J \in L(\mathbb{C}^4)$  be  $4 \times 4$  unit matrix. Thus we can perform a decomposition

$$\Gamma^\mu = \gamma^\mu \otimes \left( \bigotimes_{i=1}^{[n/2]} \sigma_1 \right)\tag{A.7}$$

or

$$\Gamma^\mu = \begin{pmatrix} 0 & \cdots & \gamma^\mu \\ & \ddots & \\ \gamma^\mu & \cdots & 0 \end{pmatrix}.\tag{A.8}$$

For  $A \neq \mu$  ( $\tilde{A} \neq \mu$ ) one gets (for  $n = 2l$  or  $n + n_1 = 2l$ )

$$\begin{aligned}\Gamma^{2p+1} &= iJ \otimes \left( \bigotimes_{i=1}^{p-2} I \right) \otimes \sigma_3 \otimes \left( \bigotimes_{i=1}^{l-p+1} \sigma_1 \right) \\ \Gamma^{2p+2} &= iJ \otimes \left( \bigotimes_{i=1}^{p-2} I \right) \otimes \sigma_3 \otimes \left( \bigotimes_{i=1}^{l-p+1} \sigma_1 \right)\end{aligned}\tag{A.9}$$

where  $4 < 2p + 1 < 2p + 2 \leq n + 4 = 2l + 2$  (or  $4 < 2p + 1 < 2p + 2 \leq n + n_1 + 4 = 2l + 2$ ). In the case  $n = 2l$  we define also a matrix

$$\Gamma^{n+5} = ii^{3(l+1)} \prod_{A=1}^{n+4} \Gamma^A = \gamma^5 \otimes \bigotimes_{i=1}^l \sigma_1 = \Gamma^{2l+5}\tag{A.10}$$

$$\Gamma^{n_1+n+5} = ii^{3(l+1)} \prod_{A=1}^{n+n_1+4} \Gamma^{\tilde{A}} = \gamma^5 \otimes \bigotimes_{i=1}^l \sigma_1 = \Gamma^{2l+5}\tag{A.11}$$

or

$$\Gamma^{n+5} = \begin{pmatrix} 0 & \cdots & \gamma^5 \\ & \ddots & \\ \gamma^5 & \cdots & 0 \end{pmatrix}\tag{A.12}$$

where  $n = 2l$ ,  $l \in N_1^\infty$ ,

$$\Gamma^{n+n_1+5} = \begin{pmatrix} 0 & \cdots & \gamma^5 \\ & \ddots & \\ \gamma^5 & \cdots & 0 \end{pmatrix},\tag{A.13}$$

$n_1 + n = 2l$ ,  $l \in N_1^\infty$ .

If  $n = 2l + 1$  we have

$$\bar{\Gamma}^A = \Gamma^A, \quad A = 1, 2, \dots, 2l + 4, \quad \bar{\Gamma}^{n+4} = \Gamma^{2l+5} = \begin{pmatrix} 0 & \cdots & \gamma^5 \\ & \ddots & \\ \gamma^5 & \cdots & 0 \end{pmatrix}.\tag{A.14}$$



Similarly, if  $n + n_1 = 2l + 1$  we have

$$\bar{T}^{\tilde{A}} = \Gamma^{\tilde{A}}, \quad \tilde{A} = 1, 2, \dots, 2l + 4, \quad \bar{T}^{n+n_1+4} = \Gamma^{2l+5} = \begin{pmatrix} 0 & \cdots & \gamma^5 \\ & \ddots & \\ \gamma^5 & \cdots & 0 \end{pmatrix}. \quad (\text{A.15})$$

It is easy to check that

$$(\Gamma^{2l+5})^2 = -1, \quad \{\tilde{\Gamma}^A, \Gamma^{2l+5}\} = 0 \quad \text{for } A \neq 2l + 5, \quad (\text{A.16})$$

$$B = \bar{B} \otimes \left( \bigotimes_{i=1}^{[n/2]} \sigma_1 \right), \quad \gamma^{\mu+} = \bar{B} \gamma^\mu \bar{B}^{-1}. \quad (\text{A.17})$$

The same we get for  $n + n_1 + 4$  case

$$\{\tilde{\Gamma}^{\tilde{A}}, \Gamma^{2l+5}\} = 0 \quad \text{for } \tilde{A} \neq 2l + 5, \quad (\text{A.18})$$

$$B = \bar{B} \otimes \left( \bigotimes_{i=1}^{[(n+n_1)/2]} \sigma_1 \right).$$

Usually  $B = \gamma^4$ .

$$\bar{T}^{21} = \begin{pmatrix} 0 & \cdots & \gamma^5 \\ & \ddots & \\ \gamma^5 & \cdots & 0 \end{pmatrix} \quad (\text{A.19})$$

$$(\bar{T}^{21})^2 = 1 \quad (\text{A.20})$$

We can proceed in a little different way:

Generalized Dirac matrices are defined by the relations

$$\{\Gamma^A, \Gamma^B\} = 2\eta^{AB} \quad \text{or} \quad \{\Gamma^{\tilde{A}}, \Gamma^{\tilde{B}}\} = 2\eta^{\tilde{A}\tilde{B}} \quad (\text{A.21})$$

where

$$\eta^{AB} = \text{diag}\{-1, -1, -1, 1, \underbrace{-1, \dots, -1}_n\} \quad (\text{A.22})$$

$$\eta^{\tilde{A}\tilde{B}} = \text{diag}\{-1, -1, -1, 1, \underbrace{-1, \dots, -1}_{n_1}, \underbrace{-1, \dots, -1}_n\}. \quad (\text{A.23})$$

For  $(n + 4)$  or  $(n + n_1 + 4)$  equal to  $2l + 2$  (the even case) we define

$$\Gamma^{4\pm} = \frac{1}{2}(\pm\Gamma^4 + \Gamma^1),$$

$$\Gamma^{\bar{A}\pm} = \frac{1}{2}(\Gamma^{2\bar{A}} \pm i\Gamma^{2\bar{A}+1}), \quad \bar{A} = 1, \dots, l. \quad (\text{A.24})$$

It is easy to show

$$\{\Gamma^{\bar{A}\pm}, \Gamma^{\bar{B}\pm}\} = -\delta^{\bar{A}\bar{B}}$$

$$\{\Gamma^{\bar{A}+}, \Gamma^{\bar{B}+}\} = \{\Gamma^{\bar{A}-}, \Gamma^{\bar{B}-}\} = 0. \quad (\text{A.25})$$

In particular

$$(\Gamma^{\bar{A}+})^2 = (\Gamma^{\bar{A}-})^2 = 0. \quad (\text{A.26})$$

In this way we always have a spinor  $\Psi_0$  such that

$$\Gamma^{\bar{A}-}\Psi_0 = 0 \quad (\text{A.27})$$

for all  $\bar{A}$ . We get all possible spinors acting on  $\Psi_0$  by  $\Gamma^{\bar{A}+}$ . We get  $2^{l+1}$  such spinors (a full representation).  $\Gamma^A$  or  $\Gamma^{\bar{A}}$  can be derived in such a base by using iterative method.

In the case of  $2l+3$  (an odd case) we should have

$$\Gamma^{2l+3} = i^{-(l+1)} \Gamma^1 \dots \Gamma^{2l+2} \quad (\text{A.28})$$

such that

$$(\Gamma^{2l+3})^2 = -1, \quad \{\Gamma^{2l+3}, \Gamma^{\bar{A}}\} = 0, \quad \bar{A} = 1, \dots, 2l+2. \quad (\text{A.29})$$

It is easy to define a basis of spinors for both cases. Let  $\zeta = (\zeta_1, \dots, \zeta_l)$ ,  $\zeta_{\bar{A}} = \pm \frac{1}{2}$ ,

$$\Psi_\zeta = \left( \prod_{\bar{A}=0}^l (\Gamma^{(l+\bar{A})})^{\zeta_{(l+\bar{A})}+1/2} \right) \Psi_0. \quad (\text{A.30})$$

$\Gamma^{2l+3}$  in the even case distinguishes between two classes of spinors

$$\begin{aligned} \Gamma^{2l+3}\Psi_\zeta &= +\Psi_\zeta \quad (2^l\text{-dimension—first representation}) \\ \Gamma^{2l+3}\Psi_\zeta &= -\Psi_\zeta \quad (2^l\text{-dimension—second representation}) \end{aligned} \quad (\text{A.31})$$

In the odd case we have only one representation of  $2^{l+1}$ -dimension.

We can introduce also generators of  $\text{SO}(1, 3+n)$  or  $\text{SO}(1, 3+n_1+n)$  algebra

$$\begin{aligned} \bar{\sigma}^{AB} \quad \text{or} \quad \bar{\sigma}^{\bar{A}\bar{B}} \\ \bar{\sigma}^{AB} &= \frac{1}{4}[\Gamma^A, \Gamma^B] \\ \bar{\sigma}^{\bar{A}\bar{B}} &= \frac{1}{4}[\Gamma^{\bar{A}}, \Gamma^{\bar{B}}]. \end{aligned} \quad (\text{A.32})$$

We have of course

$$\begin{aligned} [\bar{\sigma}^{MN}, \bar{\sigma}^{RS}] &= -[\eta^{NS}\bar{\sigma}^{MR} + \eta^{RN}\bar{\sigma}^{SM} + \eta^{MR}\bar{\sigma}^{NS} + \eta^{SM}\bar{\sigma}^{RS}] \\ [\bar{\sigma}^{\tilde{M}\tilde{N}}, \bar{\sigma}^{\tilde{R}\tilde{S}}] &= -[\eta^{\tilde{N}\tilde{S}}\bar{\sigma}^{\tilde{M}\tilde{R}} + \eta^{\tilde{R}\tilde{N}}\bar{\sigma}^{\tilde{S}\tilde{M}} + \eta^{\tilde{M}\tilde{R}}\bar{\sigma}^{\tilde{N}\tilde{S}} + \eta^{\tilde{S}\tilde{M}}\bar{\sigma}^{\tilde{R}\tilde{S}}]. \end{aligned} \quad (\text{A.33})$$

Our spinors transform as

$$\begin{aligned} \Psi &\rightarrow \exp(\frac{1}{2}\varepsilon_{AB}\hat{\sigma}^{AB})\Psi \\ \text{or } \Psi &\rightarrow \exp(\frac{1}{2}\varepsilon_{\bar{A}\bar{B}}\bar{\sigma}^{\bar{A}\bar{B}})\Psi \\ \varepsilon_{AB} &= -\varepsilon_{BA} \\ \varepsilon_{\bar{A}\bar{B}} &= -\varepsilon_{\bar{B}\bar{A}}. \end{aligned} \quad (\text{A.34})$$

We also have

$$\begin{aligned} (\bar{\sigma}^{AB})^+ \Gamma^4 &= \Gamma^4 \bar{\sigma}^{AB} \\ (\bar{\sigma}^{\bar{A}\bar{B}})^+ \Gamma^4 &= \Gamma^4 \bar{\sigma}^{\bar{A}\bar{B}}. \end{aligned} \quad (\text{A.35})$$

In our particular cases with or without spontaneous symmetry breaking we get our matrices using ordinary Dirac matrices and their tensor products with some special matrices. One gets for covariant derivatives

$$\begin{aligned}\tilde{D}\Psi &= d\Psi + \frac{1}{2}\tilde{\omega}_{AB}\bar{\sigma}^{AB}\Psi \\ \tilde{D}\Psi &= d\Psi + \frac{1}{2}\tilde{\omega}_{\tilde{A}\tilde{B}}\bar{\sigma}^{\tilde{A}\tilde{B}}\Psi.\end{aligned}\tag{A.36}$$

Moreover, we use as before (see Ref. [9])

$$\begin{aligned}\overset{\text{gauge}}{\tilde{D}}\Psi &= \text{hor}\tilde{D}\Psi = \overset{\text{gauge}}{d}\Psi + \frac{1}{2}\text{hor}(\tilde{\omega}_{AB})\bar{\sigma}^{AB}\Psi \\ \overset{\text{gauge}}{\tilde{D}}\Psi &= \text{hor}\tilde{D}\Psi = \overset{\text{gauge}}{d}\Psi + \frac{1}{2}\text{hor}(\tilde{\omega}_{\tilde{A}\tilde{B}})\bar{\sigma}^{\tilde{A}\tilde{B}}\Psi\end{aligned}\tag{A.37}$$

and also

$$\begin{aligned}\tilde{D}\bar{\Psi} &= d\bar{\Psi} - \frac{1}{2}\tilde{\omega}_{AB}\bar{\Psi}\bar{\sigma}^{AB} \\ \text{or } \tilde{D}\bar{\Psi} &= d\bar{\Psi} - \frac{1}{2}\tilde{\omega}_{\tilde{A}\tilde{B}}\bar{\Psi}\bar{\sigma}^{\tilde{A}\tilde{B}}\end{aligned}\tag{A.38}$$

where

$$\bar{\Psi} = \Psi^+ \Gamma^4\tag{A.39}$$

and similarly

$$\begin{aligned}\overset{\text{gauge}}{\tilde{D}}\bar{\Psi} &= \overset{\text{gauge}}{d}\bar{\Psi} - \frac{1}{2}\text{hor}(\tilde{\omega}_{AB})\bar{\Psi}\bar{\sigma}^{AB} \\ \text{or } \overset{\text{gauge}}{\tilde{D}}\bar{\Psi} &= \overset{\text{gauge}}{d}\bar{\Psi} - \frac{1}{2}\text{hor}(\tilde{\omega}_{\tilde{A}\tilde{B}})\bar{\Psi}\bar{\sigma}^{\tilde{A}\tilde{B}}.\end{aligned}\tag{A.40}$$

$\tilde{\omega}_{AB}$  and  $\tilde{\omega}_{\tilde{A}\tilde{B}}$  are Levi-Civita connections defined on  $P$  with respect to a symmetric part of metrics  $\gamma_{(AB)}$  and  $\gamma_{(\tilde{A}\tilde{B})}$ .

How does an iterative method for a construction of  $\Gamma$  matrices work? Let us suppose we have ordinary Dirac matrices  $\gamma^\mu$  and let us define

$$\begin{aligned}\Gamma^\mu &= \gamma^\mu \otimes \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mu = 1, 2, 3, 4, \\ \Gamma^5 &= J \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\ \Gamma^6 &= J \otimes \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad J \text{ an identity matrix, } 4 \times 4.\end{aligned}\tag{A.41}$$

Next step

$$\begin{aligned}\bar{\Gamma}^A &= \Gamma^A \otimes \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad A = 1, 2, 3, 4, 5, 6, \\ \bar{\Gamma}^7 &= I_6 \otimes I_6 \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\ \bar{\Gamma}^8 &= I_6 \otimes I_6 \otimes \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad I_6 \text{ an identity matrix, } 6 \times 6.\end{aligned}\tag{A.42}$$

For a future convenience let us apply our formalism in 20-dimensional case (i.e. for GSW model). One gets

$$\{\Gamma^A, \Gamma^B\} = 2\eta^{AB} \quad (\text{A.43})$$

$$\eta^{AB} = \text{diag}(-1, -1, -1, 1, -1, -1, \underbrace{-1, -1, \dots, -1}_{14}) \quad (\text{A.44})$$

Let us define

$$\Gamma^A = \tilde{\Gamma}^A \otimes I, \quad 1 \leq A \leq 4, \quad (\text{A.45})$$

$$\Gamma^A = \tilde{\Gamma}^5 \otimes \Gamma^{A-4}, \quad 4 < A \leq 20, \quad (\text{A.46})$$

or

$$\Gamma^\mu = \gamma^\mu \otimes I_8, \quad 1 \leq \mu \leq 4, \quad (\text{A.47})$$

$$\Gamma^5 = \gamma^5, \quad (\text{A.48})$$

$$\Gamma^{r+4} = \gamma^5 \otimes \Gamma^r, \quad 1 < r \leq 16. \quad (\text{A.49})$$

$$\overline{\Gamma}^{21} = -(-i)^9 \cdot \Gamma^1 \dots \Gamma^{20} = i \cdot \Gamma^1 \dots \Gamma^{20} \quad (\text{A.50})$$

$$\begin{aligned} \overline{\Gamma}^{21} &= \gamma^5 \otimes \overline{\Gamma}^{17} \\ \overline{\Gamma}^{17} &= (-i)^8 \Gamma^1 \dots \Gamma^{16} = \Gamma^1 \dots \Gamma^{16}, \end{aligned} \quad (\text{A.51})$$

$I_8$  — identity matrix  $8 \times 8$ ,  $\gamma^\mu$  —  $4 \times 4$  matrices as usual,  $\Gamma^A$  —  $10 \times 10$  matrices.

## Appendix B

In this appendix we give some formulas for covariant derivative used by us in the paper (see Refs [17], [18], [19], [9]). We have

$$\begin{aligned} \mathcal{D}\Psi &= \overset{\text{gauge}}{d}\Psi + \text{hor}(\tilde{\omega}_{\tilde{B}}^{\tilde{A}})\hat{\sigma}_{\tilde{A}}^{\tilde{B}}\Psi = \left( \overset{\text{gauge}}{d}\Psi + \xi_{\tilde{a}}\Psi\theta^{\tilde{a}} + \Phi_{\tilde{a}}^a X_a \Psi\theta^{\tilde{a}} \right) + \text{hor}(\tilde{\omega}_{\beta}^{\alpha}\hat{\sigma}_{\alpha}^{\beta})\Psi \\ &+ \text{hor}(\tilde{\omega}_{\tilde{b}}^{\tilde{a}}\hat{\sigma}_{\tilde{a}}^{\tilde{b}})\Psi + \text{hor}(\tilde{\omega}_b^a\hat{\sigma}_a^b)\Psi + \text{hor}(\tilde{\omega}_b^{\alpha}\hat{\sigma}_{\alpha}^b)\Psi + \text{hor}(\tilde{\omega}_{\beta}^a)\hat{\sigma}_a^{\beta}\Psi + \text{hor}(\tilde{\omega}_{\tilde{b}}^a)\hat{\sigma}_a^{\tilde{b}}\Psi, \end{aligned} \quad (\text{B.1})$$

$$\overset{\text{gauge}}{d}\overline{\Psi} = \text{hor}(d\overline{\Psi}), \quad \overset{\text{gauge}}{\overline{d}}\overline{\Psi} = \text{hor}(d\overline{\Psi}),$$

$$\begin{aligned} \mathcal{D}\overline{\Psi} &= \overset{\text{gauge}}{\overline{d}}\overline{\Psi} - \text{hor}(\tilde{\omega}_{\tilde{B}}^{\tilde{A}})\overline{\Psi}\hat{\sigma}_{\tilde{A}}^{\tilde{B}} = \left( \overset{\text{gauge}}{\overline{d}}\overline{\Psi} + \xi_{\tilde{a}}\overline{\Psi}\theta^{\tilde{a}} - \Phi_{\tilde{a}}^a \overline{\Psi} X_a \theta^{\tilde{a}} \right) - \text{hor}(\tilde{\omega}_{\beta}^{\alpha})\overline{\Psi}\hat{\sigma}_{\alpha}^{\beta} \\ &- \text{hor}(\tilde{\omega}_{\tilde{b}}^{\tilde{a}})\overline{\Psi}\hat{\sigma}_{\tilde{a}}^{\tilde{b}} - \text{hor}(\tilde{\omega}_b^a)\overline{\Psi}\hat{\sigma}_a^b - \text{hor}(\tilde{\omega}_b^{\alpha})\overline{\Psi}\hat{\sigma}_{\alpha}^b - \text{hor}(\tilde{\omega}_{\tilde{b}}^a)\overline{\Psi}\hat{\sigma}_a^{\tilde{b}} - \text{hor}(\tilde{\omega}_{\tilde{b}}^a)\overline{\Psi}\hat{\sigma}_a^{\tilde{b}}. \end{aligned} \quad (\text{B.2})$$

(B.1) and (B.2) cover both cases with and without spontaneous symmetry breaking. For GSW model in our approach we have

$$\begin{aligned} \mathcal{D}\Psi &= \overset{\text{gauge}}{d}\Psi + \text{hor}(\tilde{\omega}_{\tilde{B}}^{\tilde{A}})\hat{\sigma}_{\tilde{A}}^{\tilde{B}}\Psi = \left( \overset{\text{gauge}}{d}\Psi + \partial_5\Psi\theta^5 + \partial_6\Psi\theta^6 + \Phi_5\Psi\theta^5 + \Phi_6\Psi\theta^6 \right) \\ &+ \text{hor}(\tilde{\omega}_{\beta}^{\alpha})\hat{\sigma}_{\alpha}^{\beta}\Psi + 2\text{hor}(\tilde{\omega}_5^6)\hat{\sigma}_5^6\Psi + \text{hor}(\tilde{\omega}_b^a)\hat{\sigma}_a^b\Psi + \text{hor}(\tilde{\omega}_b^{\alpha})\hat{\sigma}_{\alpha}^b\Psi + \text{hor}(\tilde{\omega}_5^a)\hat{\sigma}_a^5\Psi \end{aligned}$$

$$+ \text{hor}(\tilde{\omega}_6^a) \hat{\sigma}_a^6 \Psi + \text{hor}(\tilde{\omega}_5^b) \hat{\sigma}_5^b \Psi + \text{hor}(\tilde{\omega}_6^b) \hat{\sigma}_6^b \Psi + \text{hor}(\tilde{\omega}_a^b) \hat{\sigma}_a^b \Psi \quad (\text{B.3})$$

$$\begin{aligned} \mathcal{D}\bar{\Psi} &= \overset{\text{gauge}}{\bar{d}} \Psi - \text{hor}(\tilde{\omega}_{\bar{B}}^{\bar{A}}) \bar{\Psi} \hat{\sigma}_{\bar{A}}^{\bar{B}} = \left( \overset{\text{gauge}}{\bar{d}} \bar{\Psi} + \partial_5 \bar{\Psi} \theta^5 + \partial_6 \bar{\Psi} \theta^6 - \bar{\Psi} \Phi_5 \theta^5 - \bar{\Psi} \Phi_6 \theta^6 \right) \\ &\quad - \text{hor}(\tilde{\omega}^{\alpha}_{\beta}) \bar{\Psi} \hat{\sigma}_{\alpha}^{\beta} - 2 \text{hor}(\tilde{\omega}_6^5) \bar{\Psi} \hat{\sigma}_5^6 - \text{hor}(\tilde{\omega}^a_{\beta}) \bar{\Psi} \hat{\sigma}_a^{\beta} - \text{hor}(\tilde{\omega}^{\alpha}_b) \bar{\Psi} \hat{\sigma}_{\alpha}^b - \text{hor}(\tilde{\omega}_5^a) \bar{\Psi} \hat{\sigma}_a^5 \\ &\quad - \text{hor}(\tilde{\omega}_6^a) \bar{\Psi} \hat{\sigma}_a^6 - \text{hor}(\tilde{\omega}_5^b) \bar{\Psi} \hat{\sigma}_5^b - \text{hor}(\tilde{\omega}_6^b) \bar{\Psi} \hat{\sigma}_6^b - \text{hor}(\tilde{\omega}_a^b) \bar{\Psi} \hat{\sigma}_a^b. \end{aligned} \quad (\text{B.4})$$

$\tilde{\omega}_{\bar{B}}^{\bar{A}}$  is a Levi-Civita connection generated by a symmetric part if a tensor  $\gamma_{(\bar{A}\bar{B})}$  on  $P$ .  $\tilde{\omega}^{\alpha}_{\beta}$  is a Levi-Civita connection generated by  $g_{(\alpha\beta)}$  on a space-time  $E$ . We have also a covariant derivative of spinors on a sphere  $S^2$ . One gets

$$\tilde{\nabla}_5 \Psi = \tilde{\nabla}_{\psi} \Psi = \frac{\partial}{\partial \psi} \Psi + \tilde{\Gamma}_6^5{}_{\psi} \hat{\sigma}_5^6 \Psi + \tilde{\Gamma}_5^6{}_{\psi} \hat{\sigma}_6^5 \Psi \quad (\text{B.5})$$

$$\tilde{\nabla}_6 \Psi = \tilde{\nabla}_{\varphi} \Psi = \frac{\partial}{\partial \varphi} \Psi + \tilde{\Gamma}_6^5{}_{\varphi} \hat{\sigma}_5^6 \Psi + \tilde{\Gamma}_6^5{}_{\varphi} \hat{\sigma}_6^5 \Psi \quad (\text{B.6})$$

$$\tilde{\nabla}_5 \bar{\Psi} = \tilde{\nabla}_{\psi} \bar{\Psi} = \frac{\partial}{\partial \psi} \bar{\Psi} - \bar{\Psi} \hat{\sigma}_5^6 \tilde{\Gamma}_6^5{}_{\psi} - \bar{\Psi} \hat{\sigma}_6^5 \tilde{\Gamma}_5^6{}_{\psi} \quad (\text{B.7})$$

$$\tilde{\nabla}_6 \bar{\Psi} = \tilde{\nabla}_{\varphi} \bar{\Psi} = \frac{\partial}{\partial \varphi} \bar{\Psi} - \bar{\Psi} \hat{\sigma}_5^6 \tilde{\Gamma}_6^5{}_{\varphi} - \bar{\Psi} \hat{\sigma}_6^5 \tilde{\Gamma}_5^6{}_{\varphi} \quad (\text{B.8})$$

$\tilde{\Gamma}_{5\varphi}^6$  and so on are Christoffel symbols for a connection on  $S^2$ . One gets

$$\begin{aligned} \tilde{\Gamma}_{66}^5 &= \sin \psi \cos \psi \\ \tilde{\Gamma}_{56}^6 &= \tilde{\Gamma}_{65}^6 = \cot \psi. \end{aligned} \quad (\text{B.9})$$

The remaining Christoffel symbols are zero. A metric tensor on  $S^2$  is defined

$$g_{(\tilde{a}\tilde{b})} = r^2 \begin{pmatrix} -1 & 0 \\ 0 & -\sin^2 \psi \end{pmatrix} \quad (\text{B.10})$$

and the inverse of  $g_{(\tilde{a}\tilde{b})}$ :

$$g^{\tilde{a}\tilde{b}} = \frac{1}{r^2} \begin{pmatrix} -1 & 0 \\ 0 & -\frac{1}{\sin^2 \psi} \end{pmatrix}.$$

## Appendix C

In this appendix we give some elements of the Atiyah–Singer index theorem (see Refs [70], [71]). The Atiyah–Singer index theorem gives an equality between two types of indexes of an elliptic operator defined on a compact manifold  $X$ . The first is known as an analytical index and the second as a topological index. An analytical index for an elliptic operator  $D$  is defined as follows

$$\text{Index}(D) = \dim \text{Ker}(D) - \dim \text{Coker}(D) = \dim \text{Ker}(D) - \dim \text{Ker}(D^+) \quad (\text{C.1})$$

where  $D^+$  is an adjoint operator for  $D$ .

$\text{Ker}(D)$  is defined as the space of solutions

$$Df = 0. \quad (\text{C.2})$$

$D$  is of course a differential operator between two smooth vector bundles  $E, F$  on a compact manifold  $X$ ,

$$D : E \rightarrow F. \quad (\text{C.3})$$

A topological index of an elliptic differential operator is given by

$$\text{Topological index}(D) = (-1)^n \langle \text{ch}(s(D)) \cdot \text{Td}(T_C X), [X] \rangle \quad (\text{C.4})$$

where  $n$  is the dimension of the manifold  $X$ ,  $s(D)$  is the symbol of the operator  $D$ ,  $\text{ch}$  is the Chern character,  $T_C X$  is the complexified tangent bundle of  $X$ , “ $\cdot$ ” is the cup product,  $[X]$  is a fundamental class of  $X$  and

$$\langle \omega, [X] \rangle = \int_X \omega. \quad (\text{C.5})$$

If the operator  $D$  is given by the formula

$$D = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha, \quad D^\alpha = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdot \frac{\partial^{\alpha_2}}{\partial x_2^{\alpha_2}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}}, \quad \alpha = (\alpha_1, \dots, \alpha_n), \quad (\text{C.6})$$

$s(D)$  is given by

$$s(D)(x, p) = \sum_{|\alpha|=m} a_\alpha(x) p^\alpha. \quad (\text{C.7})$$

In this way (C.4) is defined in pure topological terms. The index theorem states that both indexes are equal. We also have

$$\text{Index}(D) = \text{Tr}(e^{-tD^+ D}) - \text{Tr}(e^{-tD D^+}), \quad t \in \mathbb{R}. \quad (\text{C.8})$$

The differential operator  $D$  has a pseudoinverse which is a Fredholm operator.

In our case  $D$  is  $\overset{(\text{int})}{\mathcal{D}}$  defined on a compact group manifold. It means it is a Dirac operator. Due to the index theorem an analytical index is equal to a topological invariant. Thus any smooth deformation of  $D$  cannot change the value of the index. This means that any type of smooth Levi-Civita (or even metric) connection on  $X$  is not able to change the value of an index if we change

$$\partial_a \rightarrow \tilde{\nabla}_a. \quad (\text{C.9})$$

This is valid also if we introduce a nontrivial “gauge” derivative

$$\partial_a \rightarrow \overset{\text{gauge}}{\nabla}_a \quad (\text{C.10})$$

or even both derivatives at once

$$\partial_a \rightarrow \overset{\text{gauge}}{\tilde{\nabla}}_a. \quad (\text{C.11})$$

This allows us to pose a problem of chiral fermion in our theory on a stable mathematical ground which we do in the paper (see Refs [57], [58], [59], [60], [61], [62]).

For in our theory we have to do with group manifolds ( $G$ ,  $H$  or  $G2$ ) we can consider  $\hat{X}_a$  (differential operators on the manifold, i.e. left invariant vector fields) in the place of  $\partial_a$ . In this way we have to do with anholonomic frame (a group in general is non-Abelian) and in the place of  $\partial_a$  we have  $\hat{X}_a$ . The covariant derivatives  $\tilde{\nabla}_a$ ,  $\overset{\text{gauge}}{\nabla}_a$  or  $\overset{\text{gauge}}{\tilde{\nabla}}_a$  are defined in this anholonomic frame. Moreover, the zero modes condition should be redefined in the following way

$$\Gamma^a \hat{X}_a \Psi = 0. \quad (\text{C.12})$$

Thus we have to do with a smooth deformation of condition (5.4). This is true also in the case of covariant derivative written in an anholonomic frame.

Due to the index theorem this does not change a value of a topological index because it results in a smooth deformation of a differential operator. In particular, the space of  $\Phi_i$  or  $\hat{\Phi}_i$  functions will be the same. Moreover, we have a gauge condition (C.12) for our spinor fields.

## Conclusions

In the paper we consider a problem of a geometrical coupling of fermions to the bosonic unification of fundamental interactions including gravity. We get Yukawa term in the case of spontaneous symmetry breaking and Higgs' mechanism. The further research consists in placing of existing fermion families in the scheme, possibly getting some predictions of new kind of fermions and their masses and mixing angles. In the case of the Nonsymmetric Kaluza–Klein Theory we consider also a different approach (see Refs [21], [72], [31]). However, now we consider the present one as more interesting.

Let us give the following comment. The standard Kaluza–Klein Theory uses a Levi-Civita connection on a naturally metrized gauge principale fiber bundle (see Ref. [73]). The Nonsymmetric Kaluza–Klein Theory uses non-Riemannian affine connection on a nonsymmetrically metrized gauge principal fiber bundle (see Ref. [1]). Both geometries are *different*. Moreover, there is a connection between these two theories via Bohr correspondence principle because an affine connection of the Nonsymmetric Kaluza–Klein Theory contains a Levi-Civita part. In this way if the skewsymmetric part of the metric is zero, it collapses to standard Kaluza–Klein Theory (roughly speaking). The fermion fields in the standard Kaluza–Klein Theory (as in our previous papers) are coupled via new types of gauge derivatives which are horizontal parts of exterior covariant derivatives of Levi-Civita connection on a metrized gauge principal fiber bundle. We do not consider here (i.e., in standard Kaluza–Klein Theory) zero modes of Dirac operator for fermion fields on a group manifold. In the case of the Nonsymmetric Kaluza–Klein Theory we couple zero modes of the fermion fields via the same new types of gauge derivatives as in standard Kaluza–Klein Theory. Moreover, in the case of the Nonsymmetric Kaluza–Klein Theory with spontaneous symmetry breaking and Higgs' mechanism we are getting completely new results in comparison to the standard one. This is due to the more complicated structure of spontaneous symmetry breaking and Higgs' sector going to more extended Yukawa coupling for fermions. In the simplest case considered here by us (i.e. bosonic part of GSW model in the Nonsymmetric Kaluza–Klein Theory) we get Yukawa coupling for fermions and a theory with masses of  $W^\pm$ ,  $Z^0$  and Higgs' boson agreed with an experiment. This is impossible in the standard (symmetric) Kaluza–Klein Theory.

We think that this comment could give a clear statement what distinguishes the Nonsymmetric Kaluza–Klein Theory from standard Kaluza–Klein Theory, and in what respect the fermion part is different.

In 1977 we considered an idea to use a torsion of a connection defined on a fiber bundle manifold (6-dimensional, or  $(n + 5)$ -dimensional, where  $n$  is a dimension of a gauge group). The bundle has been defined over a metrized electromagnetic bundle (a bundle over a bundle). The metrization has been achieved according to the Trautman–Tulczyjew idea (see Ref. [73]). We wanted to geometrize Higgs’ field, spontaneous symmetry breaking and Higgs’ mechanism.

This idea has been developed further in order to get self-interaction terms in a scalar curvature derived for a metric connection (moreover, with non-vanishing torsion) for a Higgs’ field. The Higgs’ field has been interpreted as  $A_5$  (the fifth coordinate of an electromagnetic potential). If we suppose that  $A_5 = Q$  (a scalar field) generates a torsion of a connection on a metrized fiber bundle we get a self-interacting potential of the scalar field  $Q$ . The torsion should have a special dependence on the scalar field  $Q$  (a Higgs’ field). We considered also  $(n + 5)$ -dimensional case with a multiplet of scalar fields  $Q^a$ , where  $Q^a = A_5^a$  is a part of a gauge field over additional dimensions. We abandon this idea as useless and do not develop it further. Moreover, an idea of  $A_5$ ,  $A_5^a$  as Higgs’ fields is correct and it was developed in our future papers.

This is a historical remark.

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<sup>1</sup>Mathematica<sup>TM</sup> is the registered mark of Wolfram Co.



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